KÄHLER-RICCI FLOW WITH UNBOUNDED CURVATURE

SHAOCHUANG HUANG AND LUEN-FAI TAM[1]

Abstract. Let $g(t)$ be a complete solution to the Ricci flow on a noncompact manifold such that $g(0)$ is Kähler . We prove that if $|\mathrm{Rm}(g(t))|_{g(t)} \le a/t$ for some $a > 0$, then $g(t)$ is Kähler for $t > 0$. We prove that there is a constant $a(n) > 0$ depending only on n such that the following is true: Suppose $g(t)$ is a complete solution to the Kähler-Ricci flow on a noncompact n-dimensional complex manifold such that $g(0)$ has nonnegative holomorphic bisectional curvature and such that $|\mathrm{Rm}(g(t))|_{g(t)} \le a(n)/t$, then $g(t)$ has nonnegative holomorphic bisectional curvature for $t > 0$. These generalize the results in [21]. As corollaries, we prove that (i) any complete noncompact Kähler manifold with nonnegative complex sectional curvature with maximum volume growth is biholomorphic to $\mathbb{C}^n$; and (ii) there is $\epsilon(n) > 0$ depending only on n such that if (M^n, g_0) is a complete noncompact Kähler manifold of complex dimension n with nonnegative holomorphic bisectional curvature and maximum volume growth and if $(1+\epsilon(n))^{-1}h \le g_0 \le (1+\epsilon(n))h$ for some Riemannian metric h with bounded curvature, then M is biholomorphic to $\mathbb{C}^n$.

Keywords: Ricci flow, Kähler condition, holomorphic bisectional curvature, uniformization

1. Introduction

In [18], Simon proved that there is a constant $\epsilon(n) > 0$ depending only on n such that if (M^n, g_0) is a complete n dimensional Riemannian manifold and if there is another metric h with curvature bounded by k_0 with

$$(1+\epsilon(n))^{-1}h \le g_0 \le (1+\epsilon(n))h,$$

then the so-called h-flow has a short time solution $g(t)$ such that

$$|\mathrm{Rm}(g(t))|_{g(t)} \le C/t. \tag{1.1}$$

Here h-flow is basically the usual Ricci-DeTurck flow. If $h = g_0$, the h-flow is exactly the Ricci-DeTurck flow. For the precise definition of h-flow, see Section 5. It is not hard to construct Ricci flow using the solution of h-flow if g_0 is smooth. On the other hand, in [2], Cabezas-Rivas and Wilking proved that if (M, g_0) is a complete noncompact Riemannian manifold with nonnegative complex sectional curvature, and if the volume of geodesic ball $B(x,1)$ of

Date: June, 2015.

2010 *Mathematics Subject Classification.* Primary 32Q15; Secondary 53C44 .

[1]Research partially supported by Hong Kong RGC General Research Fund #CUHK 1430514.

radius 1 with center at x is uniformly bounded below away from 0, then the Ricci flow have a solution for short time with nonnegative complex sectional curvature so that (1.1) holds. Recall that a Riemannian manifold is said to have nonnegative complex sectional curvature if $R(X,Y,\bar{Y},\bar{X}) \geq 0$ for any vectors in the complexified tangent bundle.

It is natural to ask the following:

Question: Suppose g_0 is Kähler. Are the above solutions $g(t)$ of Ricci flow also Kähler for $t > 0$?

This question has been studied before. It was proved by Yang and Zheng [24] for a $U(n)$ invariant initial Kähler metric on $\mathbb{C}^n$, the solution constructed by Cabezas-Rivas and Wilking is Kähler for $t > 0$, under some additional technical conditions.

It is well-known that if M is compact or if the curvature of g_0 is bounded, the answer to the above question is yes by [11] and [21]. In this paper, we want to prove the following:

Theorem 1.1. *If (M^n, g_0) is a complete noncompact Kähler manifold with complex dimension n and if $g(t)$ is a smooth complete solution to the Ricci flow on $M \times [0,T]$, $T > 0$, with $g(0) = g_0$ such that*

$$|\mathrm{Rm}(g(t))|_{g(t)} \leq \frac{a}{t}$$

for some $a > 0$, then $g(t)$ is Kähler for all $0 \leq t \leq T$.

This gives an affirmative answer to the above question. The result is related to previous works on the on the existence of Kähler-Ricci flows without curvature bound, see [3, 4, 10, 24] for example.

We may apply the theorem to the uniformization conjecture by Yau [25] which states that a complete noncompact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{C}^n$. A previous result by Chau and the second author [5] says that the conjecture is true if the Kähler manifold has maximum volume growth and has *bounded curvature*, see also [14, 7]. Combining this with the theorem, we have:

Corollary 1.1. *Let (M^n, g_0) be a complete noncompact Kähler manifold with complex dimension n and with nonnegative complex sectional curvature. Suppose M^n has maximum volume growth. Then M^n is biholomorphic to $\mathbb{C}^n$.*

For Kähler surface, sectional curvature being nonnegative is equivalent to complex sectional curvature being nonnegative [26]. Hence in particular, any complete Kähler surface with nonnegative sectional curvature with maximum volume growth is biholomorphic to $\mathbb{C}^2$. We should mention that recently Liu [12] proves that a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature and with maximum volume growth is biholomorphic to an affine algebraic variety, generalizing the result of Mok [14]. Moreover, if the volume of geodesic balls are close to the Euclidean balls with same radii, then the manifold is biholomorphic to $\mathbb{C}^n$.

By Theorem 1.1, we know that from the solution constructed by Simon [18] one can construct a solution to the Kähler-Ricci flow if g_0 is Kähler . In view of the conjecture of Yau, we would like to know that if the nonnegativity of holomorphic bisectional curvature will be preserved by the solution $g(t)$ of the Kähler-Ricci flow. The second result in this paper is the following:

Theorem 1.2. *There is $0 < a(n) < 1$ depending only on n such that if $g(t)$ is a complete solution of Kähler-Ricci flow on $M \times [0, T]$ with $|\mathrm{Rm}(g(t))|_{g(t)} \leq \frac{a}{t}$, where M is an n-dimensional non-compact complex manifold. If $g(0)$ has nonnegative holomorphic bisectional curvature, then so does $g(t)$ for all $t \in [0, T]$.*

We should mention that in [24], Yang and Zheng proved that the nonnegativity of bisectional curvature is preserved under the Kähler-Ricci flow for $U(n)$ invariant solution on $\mathbb{C}^n$ without any condition on the bound of the curvature.

By refining the estimates in [18], one can prove that if $\epsilon(n) > 0$ is small in the result of Simon, then curvature of the solution of the h-flow will be bounded by a/t with a small. Hence as a corollary to the theorem, using [5] again, we have:

Corollary 1.2. *There is $\epsilon(n) > 0$, depending only on n. Suppose (M^n, g_0) is a complete noncompact Kähler manifold with complex dimension n with nonnegative holomorphic bisectional curvature with maximum volume growth. Suppose there is a Riemannian metric h on M with bounded curvature such that $(1 + \epsilon(n))^{-1}h \leq g_0 \leq (1 + \epsilon(n))h$. Then M is biholomorphic to $\mathbb{C}^n$.*

By a result of Xu [23], we also have the following corollary which says that the condition that the curvature is bounded in the uniformization result in [5] can be relaxed to the condition that the curvature is bounded in some integral sense. Namely, we have:

Corollary 1.3. *Let (M^n, g_0) is a complete noncompact Kähler manifold with complex dimension $n \geq 2$ with nonnegative holomorphic bisectional curvature with maximum volume growth. Suppose there is $r_0 > 0$ and there is $C > 0$ such that*

$$\left(\fint_{B_x(r_0)} |\mathrm{Rm}|^p \right)^{\frac{1}{p}} \leq C$$

for some $p > n$ for all $x \in M$. Then M is biholomorphic to $\mathbb{C}^n$.

The paper is organized as follows: in Section 2 we prove a maximum principle and apply it in Section 3 to prove Theorem 1.1. In Section 4 we prove Theorem 1.2. In Section 5, we will construct solution to the Kähler-Ricci flow with nonnegative holomorphic bisectional curvature through the h-flow.

Acknowledgement: The second author would like to thank Albert Chau for some usefully discussions and for bringing our attention to the results in [23].

2. A maximum principle

In this section, we will prove a maximum principle, which will be used in the proof of Theorem 1.1.

Let (M^n, g_0) be a complete noncompact Riemannian manifold. Let $g(t)$ be a smooth complete solution to the Ricci flow on $M \times [0,T]$, $T > 0$ with $g(0) = g_0$, i.e.

$$\begin{cases} \frac{\partial}{\partial t} g &= -2\mathrm{Ric}, \quad \text{on } M \times [0,T]; \\ g(0) = g_0. \end{cases} \tag{2.1}$$

Let Γ and $\bar{\Gamma}$ be the Christoffel symbols of $g(t)$ and $\bar{g} = g(T)$ respectively. Let $A = \Gamma - \bar{\Gamma}$. Then A is a $(1,2)$ tensor. In the following, lower case $c, c_1, c_2, \ldots$ will denote positive constants depending only on n.

Lemma 2.1. *With the above notation and assumptions, suppose the curvature satisfies* $|\mathrm{Rm}(g(t)|_{g(t)} \le at^{-1}$ *for some positive constant* a. *Then there is a constant* $c = c(n) > 0$, *such that*

(i)
$$\left(\frac{T}{t}\right)^{-ca} \bar{g} \le g(t) \le \left(\frac{T}{t}\right)^{ca} \bar{g};$$

(ii) $|\nabla \mathrm{Rm}| \le Ct^{-\frac{3}{2}}$ *for some constant* $C = C(n,T,a) > 0$ *depending only on* n, T, a;

(iii)
$$|A|_{\bar{g}} \le Ct^{-\frac{1}{2}-ca},$$
for some constant $C = C(n,T,a) > 0$ *depending only on* n, T *and* a.

Proof. (i) follows from the Ricci flow equation.

(ii) is a result in [20], see also [9, Theorem 7.1].

To prove (iii), in local coordinates:

$$\frac{\partial}{\partial t} A_{ij}^k = -g^{kl} \left(\nabla_i R_{jl} + \nabla_j R_{il} - \nabla_l R_{ij} \right).$$

At a point where $\bar{g}_{ij} = \delta_{ij}$ such that $g_{ij} = \lambda_i \delta_{ij}$

$$\begin{aligned} \left| \frac{\partial}{\partial t} |A|_{\bar{g}}^2 \right| \le & C_1(n,T) t^{-c_1 a} |\nabla \mathrm{Ric}|_{\bar{g}} |A|_{\bar{g}} \\ \le & C_2(n,T,a) t^{-c_2 a - \frac{3}{2}} |A|_{\bar{g}} \end{aligned}$$

for some constants C_1, C_2 depending only on n, T, a and c_1, c_2 depending only on n. From this the result follows. □

Under the assumption of the lemma, since $g(T)$ is complete and the curvature of $\bar{g} = g(T)$ is bounded by a/T, we can find a smooth function ρ on M such that

$$d_{\bar{g}}(x, x_0) + 1 \le \rho(x) \le C'(d(x,x_0) + 1); \ |\bar{\nabla} \rho|_{\bar{g}} + |\bar{\nabla}^2 \rho|_{\bar{g}} \le C' \tag{2.2}$$

for some $C' > 1$, where $\bar\nabla$ is covariant derivative with respect to $\bar g$ and $C' > 0$ is a constant depending on n and a/T, see [21, 22].

Lemma 2.2. *With the same assumptions and notation as in the previous lemma, $\rho(x)$ satisfies*

$$|\nabla\rho| \le C_1 t^{-ca}$$

and

$$|\Delta\rho| \le C_2 t^{-\frac{1}{2}-ca}$$

where C_1, C_2 depending only on n, T, a and $c > 0$ depending only on n. Here ∇ and Δ are the covariant derivative and Laplacian of $g(t)$ respectively.

Proof. The first inequality follows from Lemma 2.1(i). To estimate $\Delta\rho$, at a point where $\bar g_{ij} = \delta_{ij}$ and g_{ij} is diagonalized, we have

$$\begin{aligned}
\left|\Delta\rho - \bar\Delta\rho\right| &= \left|g^{ij}\nabla_i\nabla_j\rho - g_T^{ij}\overline{\nabla}_i\overline{\nabla}_j\rho\right| \\
&\le \left|g^{ij}\left(\nabla_i\nabla_j - \overline{\nabla}_i\overline{\nabla}_j\right)\rho\right| + \left|\left(g^{ij} - g_T^{ij}\right)\overline{\nabla}_i\overline{\nabla}_j\rho\right| \\
&\le |g^{ij}A_{ij}^k\rho_k| + C_3 t^{-c_1 a} \\
&\le C_4 t^{-\frac{1}{2}-c_2 a}
\end{aligned}$$

for some constants C_3, C_4 depending only on n, T, a, and c_1, c_2 depending only on n. By the estimates of $\bar\Delta\rho$, the second result follows. □

Lemma 2.3. *Let (M^n, g) be a complete noncompact Riemannian manifold with dimension n and let $g(t)$ be a smooth complete solution of the Ricci flow on $M \times [0, T]$, $T > 0$ such that the curvature satisfies $|\mathrm{Rm}| \le at^{-1}$ for some $a > 0$.*

Let $f \ge 0$ be a smooth function on $M \times [0, T]$ such that

(i)

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \le \frac{a}{t} f;$$

(ii) $\frac{\partial^k f}{\partial t^k}|_{t=0} = 0$ *for all $k \ge 0$;*

(iii) $\sup_{x\in M} f(x,t) \le Ct^{-l}$, *for some positive integer l for some constant C.*

Then $f \equiv 0$ on $M \times [0, T]$.

Proof. We may assume that $T \le 1$. In fact, if we can prove that $f \equiv 0$ on $M \times [0, T_1]$ where $T_1 = \min\{1, T\}$, then it is easy to see that $f \equiv 0$ on $M \times [0, T]$ because f and the curvature of $g(t)$ are uniformly bounded on $M \times [T_1, T]$.

Let $p \in M$ be a fixed point, and let $d(x, t)$ be the distance between p, x with respect to $g(t)$. By [16] (see also [8, Chapter 18]), for all r_0, if $d(x, t) > r_0$, then

$$\frac{\partial_-}{\partial t} d(x,t) - \Delta_t d(x,t) \ge -C_0\left(t^{-1}r_0 + \frac{1}{r_0}\right) \tag{2.3}$$

in the barrier sense, for some $C_0 = C_0(n, a)$ depending only on n and a. Here

$$\frac{\partial_-}{\partial t} d(x,t) = \liminf_{h\to 0^+} \frac{d(x,t) - d(x,t-h)}{h}. \tag{2.4}$$

The above inequality means that for any $\epsilon > 0$, there is a smooth function $\sigma(y)$ near x such that $\sigma(x) = d(x,t)$, $\sigma(y) \ge d(y,t)$ near x, such that σ is C^2 and

$$\frac{\partial_-}{\partial t} d(x,t) - \Delta_t \sigma(x) \ge -C_0 \left(t^{-1} r_0 + \frac{1}{r_0} \right) - \epsilon. \tag{2.5}$$

In the following, we always take $\epsilon = T^{-\frac{1}{2}}$.

Let f be as in the lemma. First we want to prove that for any integer $k > 0$ there is a constant B_k such that

$$\sup_{x\in M} f(x,t) \le B_k t^k. \tag{2.6}$$

We may assume that $k > a$. Let $F = t^{-k} f$, then

$$\left(\frac{\partial}{\partial t} - \Delta \right) F \le -\frac{k-a}{t} F \le 0. \tag{2.7}$$

Let $1 \ge \phi \ge 0$ be a smooth function on $[0, \infty)$ such that

$$\phi(s) = \begin{cases} 1, & \text{if } 0 \le s \le 1; \\ 0, & \text{if } s \ge 2, \end{cases}$$

and such that $-C_1 \le \phi' \le 0$, $|\phi''| \le C_1$ for some $C_1 > 0$. Let $\Phi = \phi^m$, where $m > 2$ will be chosen later. Then $\Phi = 1$ on $[0, 1]$ and $\Phi = 0$ on $[2, \infty)$, $1 \ge \Phi \ge 0$, $-C(m)\Phi^q \le \Phi' \le 0$, $|\Phi''| \le C(m)\Phi^q$. where $C(m) > 0$ depends on m and C_1, and $q = 1 - \frac{2}{m}$.

For any $r >> 1$, let $\Psi(x,t) = \Phi(\frac{d(x,t)}{r})$. Let

$$\theta(t) = \exp(-\alpha t^{1-\beta}),$$

where $\alpha > 0$, $0 < \beta < 1$ will be chosen later.

We claim that one can choose m, α and β such that for all $r >> 1$

$$H(x,t) = \theta(t)\Psi(x,t)F \le C_2$$

on $M \times [0, T]$, where C_2 is independent of r. If the claim is true, then we have F is bounded. Hence $f(x,t) \le B_k t^k$.

First note that $\Psi(x,t)$ has compact support in $M \times [0, T]$. By assumption (ii) and the fact f is smooth, we conclude that $H(x,t)$ is continuous on $M \times [0, T]$. Moreover, by (ii) again, $H(x, 0) = 0$. Suppose $H(x,t)$ attains a positive maximum at (x_0, t_0) for some $x_0 \in M$, $t_0 > 0$. Suppose $d(x_0, t_0) < r$, then there is a neighborhood U of x and $\delta > 0$ such that $d(x,t) < r$ for $x \in U$ and

$|t-t_0|<\delta$. For such (x,t), $H(x,t)=\theta(t)F(x,t)$. Since $H(x_0,t_0)$ is a local maximum, we have

$$\begin{aligned}0\le&\left(\frac{\partial}{\partial t}-\Delta\right)H\\ =&\theta(t)\left(\theta' F+\left(\frac{\partial}{\partial t}-\Delta\right)F\right)\\ <&0\end{aligned}$$

which is a contradiction.

Hence we must have $d(x_0,t_0)\ge r$. If $r>>1$, then $r\ge T^{\frac12}$, and at (x_0,t_0)

$$\frac{\partial_-}{\partial t}d(x,t)-\Delta_t d(x,t)\ge -(2C_0+1)t^{-\frac12}$$

in the barrier sense, by taking $r_0=t^{\frac12}$. Let $\sigma(x)$ be a barrier function near x_0. Let $\widetilde\Psi(x)=\Phi(\frac{\sigma(x)}{r})$, and let

$$\tilde H(x,t)=\theta(t)\tilde\Psi(x)F(x,t)$$

which is defined near x_0 for all t. Moreover,

$$\tilde H(x_0,t_0)=H(x_0,t_0)$$

and

$$\tilde H(x,t_0)\le H(x,t_0)$$

near x_0 because $\sigma(x)\ge d(x,t_0)$ near x_0 and $\Phi'\le 0$. Hence $\widetilde H(x,t_0)$ has a local maximum at (x_0,t_0) as a function of x. So we have

$$\nabla\tilde H(x_0,t_0)=0 \tag{2.8}$$

and

$$\Delta\tilde H(x_0,t_0)\le 0. \tag{2.9}$$

At (x_0,t_0)

$$\begin{aligned}0\ge&\Delta\left(\theta(t)\widetilde\Psi(x)F(x,t)\right)\\ =&\theta\widetilde\Psi\Delta F+\theta F\Delta\widetilde\Psi+2\theta\langle\nabla F,\nabla\widetilde\Psi\rangle\\ =&\theta\widetilde\Psi\Delta F+\theta F\left(\frac1r\Phi'\Delta\sigma+\frac1{r^2}\Phi''|\nabla\sigma|^2\right)-2\theta\frac{|\nabla\widetilde\Psi|^2}{\widetilde\Psi}F\\ \ge&\theta\Phi\Delta F+\theta F\left(\frac1r\Phi'\Delta\sigma+\frac1{r^2}\Phi''|\nabla\sigma|^2\right)-\frac{2}{r^2}\theta\frac{\Phi'^2}{\Phi}F\end{aligned} \tag{2.10}$$

where we have used the fact that $\sigma(x)\ge d(x,t_0)$ near x_0 and $\sigma(x_0)=d(x_0,t_0)$ so that $|\nabla\sigma(x_0)|\le 1$. Φ and the derivatives Φ' and Φ'' are evaluated at $\frac{d(x_0,t_0)}{r}$.

On the other hand,

$$
\begin{aligned}
0 \leq &\liminf_{h\to 0^+} \frac{H(x_0,t_0)-H(x_0,t_0-h)}{h}\\
=&\theta'\Psi F+\theta\Psi\frac{\partial}{\partial t}F+\theta F\liminf_{h\to 0^+}\frac{-\Psi(x_0,t_0-h)+\Psi(x_0,t_0)}{h}.
\end{aligned}
$$

Now

$$
\begin{aligned}
-\Psi(x_0,t_0-h)+\Psi(x_0,t_0)=&-\Phi(\frac{d(x_0,t_0-h)}{r})+\Phi(\frac{d(x_0,t_0)}{r})\\
=&\frac{1}{r}\Phi'(\xi)(d(x_0,t_0)-d(x_0,t_0-h)),
\end{aligned}
$$

for some ξ between $\frac{1}{r}d(x_0,t_0-h)$ and $\frac{1}{r}d(x_0,t_0)$ which implies

$$
\begin{aligned}
\liminf_{h\to 0^+}\frac{-\Psi(x_0,t_0-h)+\Psi(x_0,t_0)}{h}\leq&\limsup_{h\to 0^+}\frac{-\Psi(x_0,t_0-h)+\Psi(x_0,t_0)}{h}\\
=&\frac{1}{r}\Phi'\frac{\partial_-}{\partial t}d(x_0,t)|_{t=t_0}
\end{aligned}
$$

because $\Phi'\leq 0$, where Φ' is evaluated at $\frac{1}{r}d(x_0,t_0)$. In the following, C_i will denote positive constants independent of α,β. Combining the above inequality with (2.10), we have at (x_0,t_0):

$$
\begin{aligned}
0\leq&\theta'\Phi F+\theta\Phi\frac{\partial}{\partial t}F+\theta F\frac{1}{r}\Phi'\frac{\partial_-}{\partial t}d(x_0,t_0)\\
&-\theta\Psi\Delta F-\theta F\left(\frac{1}{r}\Phi'\Delta\sigma+\frac{1}{r^2}\Phi''|\nabla\sigma|^2\right)+\frac{2}{r^2}\theta\frac{\Phi'^2}{\Phi}F\\
\leq&\theta'\Phi F+C_2\left(t_0^{-\frac12}\Phi^q+\Phi^{2q-1}\right)\theta F\\
\leq&-\alpha(1-\beta)t_0^{-\beta}\theta\Phi F+C_3\theta\left[t_0^{-\frac12}t_0^{-(1-q)(k+l)}(\Phi F)^q+t_0^{-\frac12}t_0^{-2(1-q)(k+l)}(\Phi F)^{2q-1}\right]\\
\leq&\theta\left[-\alpha(1-\beta)t_0^{-\beta}\Phi F+C_4t_0^{-\frac12-2(1-q)(k+l)}\left((\Phi F)^q+(\Phi F)^{2q-1}\right)\right]
\end{aligned}
$$

where Φ,Φ',Φ'' are evaluated at $d(x_0,t_0)/r$. Now first choose m large enough depending only on k,l so that $\frac12+2(1-q)(k+l)=\beta<1$. Then choose α such that $\alpha(1-\beta)>2C_4$. Then one can see that we must have $\Phi F\leq 1$. Hence $H=\theta\Phi F\leq C$ at the maximum point of $H(x,t)$, where C is a constant independent of r. This completes the proof of the claim.

Next, let $F=t^{-a}f$. Then

$$
\left(\frac{\partial}{\partial t}-\Delta\right)F\leq 0.
$$

Let ρ be the function in Lemma 2.2, we have

$$
|\Delta\rho|\leq C_5t^{-b}
$$

for some $b>1$. Let $\eta(x,t)=\rho(x)\exp(\frac{2C_5}{1-b}t^{1-b})$. Note that $\eta(x,0)=0$.

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\eta =&\exp(\frac{2C_5}{1-b}t^{1-b})\left(2C_5t^{-b}\rho-\Delta\rho\right)\\ \geq& C_5t^{-b}\exp(\frac{2C_5}{1-b}t^{1-b})\\ >&0.\end{aligned}$$

where we have used the fact that $\rho \geq 1$. Since $F \leq C_6t^2$ in $M\times[0,T]$. In particular it is bounded. Then for any $\epsilon > 0$

$$\left(\frac{\partial}{\partial t}-\Delta\right)(F-\epsilon\eta-\epsilon t) < 0.$$

There is $t_1 > 0$ depending only on ϵ, C_6 such that $F - \epsilon t < 0$ for $t \leq t_1$. For $t \geq t_1$, $F - \epsilon\eta < 0$ outside some compact set. Hence if $F - \epsilon\eta - \epsilon t > 0$ somewhere, then there exist $x_0 \in M$, $t_0 > 0$ such that $F - \epsilon\eta - \epsilon t$ attains maximum. But this is impossible. So $F - \epsilon\eta - \epsilon t \leq 0$. Let $\epsilon \to 0$, we have $F = 0$. □

3. preservation of the Kähler condition

In this section, we want to prove Theorem 1.1 and give some applications. Recall Theorem 1.1 as follows:

Theorem 3.1. *If (M^n, g_0) is a complete noncompact Kähler manifold with complex dimension n and if $g(t)$ is a smooth complete solution to the Ricci flow (2.1) on $M\times[0,T]$, $T > 0$, with $g(0) = g_0$ such that*

$$|\mathrm{Rm}(g(t))|_{g(t)} \leq \frac{a}{t}$$

for some $a > 0$, then $g(t)$ is Kähler for all $0 \leq t \leq T$.

We will use the setup as in [21, Section 5]. Let $T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $T_{\mathbb{R}}M$, where $T_{\mathbb{R}}M$ is the real tangent bundle. Similarly, let $T^*_{\mathbb{C}}M = T^*_{\mathbb{R}}(M)\otimes_{\mathbb{R}}\mathbb{C}$, where $T^*_{\mathbb{R}}M$ is the real cotangent bundle. Let $z = \{z^1, z^2, \cdots, z^n\}$ be a local holomorphic coordinate on M, and

$$\begin{cases} z^k = x^k + \sqrt{-1}x^{k+n} \\ x^k \in \mathbb{R}, x^{k+n} \in \mathbb{R}, \qquad k = 1, 2, \cdots, n.\end{cases}$$

In the following:

- $i, j, k, l, \cdots$ denote the indices corresponding to real vectors or real covectors;
- $\alpha, \beta, \gamma, \delta, \cdots$ denote the indices corresponding to holomorphic vectors or holomorphic covectors,
- $A, B, C, D, \cdots$ denote both $\alpha, \beta, \gamma, \delta, \cdots$ and $\bar\alpha, \bar\beta, \bar\gamma, \bar\delta, \cdots$.

Extend $g_{ij}(t)$, $R_{ijkl}(t)$ etc. $\mathbb{C}$-linearly to the complexified bundles. We have:

$$\overline{g_{AB}} = g_{\bar{A}\bar{B}}, \quad \overline{R_{ABCD}} = R_{\bar{A}\bar{B}\bar{C}\bar{D}}.$$

In our convention, $R_{1221} = R(e_1, e_2, e_2, e_1)$ is the sectional curvature of the two-plane spanned by orthonormal pair e_1, e_2. R_{ABCD} has the same symmetry as R_{ijkl} and it satisfies the Binachi identities.

Let $g^{AB} := (g^{-1})^{AB}$, it means $g^{AB}g_{BC} = \delta^A_C$, and let

$$R_{AB} = g^{CD}R_{ACDB}$$

on $M \times [0, T]$. Then we have

$$\frac{\partial}{\partial t} g_{AB} = -2R_{AB} \tag{3.1}$$

and

(3.2)

$$\begin{aligned}\frac{\partial}{\partial t} R_{ABCD} =& \triangle R_{ABCD} - 2g^{EF}g^{GH}R_{EABG}R_{FHCD} - 2g^{EF}g^{GH}R_{EAGD}R_{FBHC}\\ &+ 2g^{EF}g^{GH}R_{EAGC}R_{FBHD} - g^{EF}(R_{EBCD}R_{FA} + R_{AECD}R_{FB}\\ &+ R_{ABED}R_{FC} + R_{ABCE}R_{FD})\end{aligned}$$

on $M \times [0, T]$.

We begin with the following lemma:

Lemma 3.1. *Let (M, g_0) be a Kähler manifold, and $g(t)$ be a smooth solution to the Ricci flow with $g(0) = g_0$. In the above set up, we have*

$$\frac{\partial^k}{\partial t^k} R_{AB\gamma\delta}|_{t=0} = 0$$

at each point of M and for all $k \geq 0$ and for all A, B, γ, δ.

Proof. Let $p \in M$ with holomorphic local coordinate z. In the following, all computations are at $(z, 0)$ unless we have emphasis otherwise. We will prove the lemma by induction. Consider the following statement:

$$H(k)\begin{cases} H_1(k) : \frac{\partial^k}{\partial t^k} R_{AB\gamma\delta} = 0\\ H_2(k) : \frac{\partial^k}{\partial t^k} g_{AB} = 0 \text{ if } A, B \text{ are of the same type}\\ H_3(k) : \frac{\partial^k}{\partial t^k} R_{AB} = 0 \text{ if } A, B \text{ are of the same type}\\ H_4(k) : \frac{\partial^k}{\partial t^k} \Gamma^C_{AB} = 0 \text{ unless } A, B, C \text{ are of the same type}\\ H_5(k) : \frac{\partial^k}{\partial t^k} R_{AB\gamma\delta;E} = 0\\ H_6(k) : \frac{\partial^k}{\partial t^k} R_{AB\gamma\delta;EF} = 0\\ H_7(k) : \frac{\partial^k}{\partial t^k} \triangle R_{AB\gamma\delta} = 0\end{cases}$$

Here we denote covariant derivative with respect to $g(t)$ by "$;$" and the partial derivative by "$,$". If $H_i(k)$ are true for all $i = 1, \cdots, 7$, we will say that $H(k)$ holds. As usual:

$$\Gamma^C_{AB} = \frac{1}{2} g^{CD} (g_{AD,B} + g_{DB,A} - g_{AB,D}).$$

We now consider the case that $k = 0$. Since the initial metric is K̈hler, it is easy to see that $H(0)$ holds. Now we assume $H(i)$ holds for all $i = 0, 1, 2, \cdots, k$. We want to show $H(k+1)$ holds. We first see that

$$\begin{aligned}
&\frac{\partial^{k+1}}{\partial t^{k+1}} R_{AB\gamma\delta} \\
=&\frac{\partial^k}{\partial t^k}(\triangle R_{AB\gamma\delta}) - \sum_{\substack{m+n+p+q=k \\ 0\le m,n,p,q\le k}} 2(g^{EF})_m (g^{GH})_n (R_{EABG})_p (R_{FH\gamma\delta})_q \\
&- \sum_{\substack{m+n+p+q=k \\ 0\le m,n,p,q\le k}} 2(g^{EF})_m (g^{GH})_n (R_{EAG\delta})_p (R_{FBH\gamma})_q \\
&+ \sum_{\substack{m+n+p+q=k \\ 0\le m,n,p,q\le k}} 2(g^{EF})_m (g^{GH})_n (R_{EAG\gamma})_p (R_{FBH\delta})_q \\
&- \sum_{\substack{m+n+p=k \\ 0\le m,n,p\le k}} (g^{EF})_m (R_{AF})_n (R_{EB\gamma\delta})_p - \sum_{\substack{m+n+p=k \\ 0\le m,n,p\le k}} (g^{EF})_m (R_{BF})_n (R_{AE\gamma\delta})_p \\
&- \sum_{\substack{m+n+p=k \\ 0\le m,n,p\le k}} (g^{EF})_m (R_{\gamma F})_n (R_{ABE\delta})_p - \sum_{\substack{m+n+p=k \\ 0\le m,n,p=\le k}} (g^{EF})_m (R_{\delta F})_n (R_{AB\gamma E})_p.
\end{aligned}$$

Here $(\,\cdot\,)_p = \frac{\partial^p}{\partial t^p}(\,\cdot\,)$.

Suppose $(g_{AB})_p = 0$ at $t = 0$ if A, B are of the same type for $p = 0, 1, \ldots, k$, then it is also true that $(g^{AB})_p = 0$ if A, B are of the same type for $p = 0, 1, \ldots, k$. On the other hand, in the RHS of the above inequality, the derivative of each term with respect to t is only up to order k, by the induction hypothesis, $H_1(k+1)$ holds. Now

$$\frac{\partial}{\partial t} g_{AB} = -2R_{AB},$$

it is easy to see that $H_2(k+1)$ holds because $H_3(k)$ holds.

Since

$$\frac{\partial^{k+1}}{\partial t^{k+1}} R_{\alpha\beta} = \sum_{\substack{m+n=k+1 \\ 0\le m,n\le k+1}} (g^{CD})_m (R_{\alpha CD\beta})_n,$$

and since that $H_1(k+1)$ and $H_2(k+1)$ hold, we conclude that $H_3(k+1)$ holds. Here we have used the symmetries of R_{ABCD}.

Since

$$\begin{aligned}\frac{\partial^{k+1}}{\partial t^{k+1}}\Gamma^{\alpha}_{A\bar\beta}=&-\sum_{\substack{m+n=k\\0\le m,n\le k}}(g^{\alpha D})_m(R_{\bar\beta D;A}+R_{AD;\bar\beta}-R_{A\bar\beta;D})_n\\=&-\sum_{\substack{m+n=k\\0\le m,n\le k}}(g^{\alpha\bar\sigma})_m(R_{\bar\beta\bar\sigma;A}+R_{A\bar\sigma;\bar\beta}-R_{A\bar\beta;\bar\sigma})_n,\end{aligned}$$

by the induction hypothesis. If $A=\bar\gamma$, then each term on the RHS is zero by the induction hypothesis. If $A=\gamma$, then

$$(R_{\bar\beta\bar\sigma;\gamma})_n=(R_{\bar\beta\bar\sigma,\gamma})_n-(\Gamma^E_{\gamma\bar\sigma}R_{E\bar\beta})_n-(\Gamma^E_{\gamma\bar\beta}R_{E\bar\sigma})_n,$$

so it vanishes because $n\le k$. On the other hand,

$$\begin{aligned}&R_{\gamma\bar\sigma;\bar\beta}-R_{\gamma\bar\beta;\bar\sigma}\\=&g^{CD}(R_{\gamma CD\bar\sigma;\bar\beta}-R_{\gamma CD\bar\beta;\bar\sigma})\\=&g^{CD}(R_{\gamma CD\bar\sigma;\bar\beta}+R_{\gamma C\bar\sigma D;\bar\beta}+R_{\gamma C\bar\beta\bar\sigma;D})\\=&g^{CD}R_{\gamma C\bar\beta\bar\sigma;D}.\end{aligned}$$

So

$$\left(R_{\gamma\bar\sigma;\bar\beta}-R_{\gamma\bar\beta;\bar\sigma}\right)_n=0$$

for $n\le k$ by the induction hypothesis. Thus,

$$\frac{\partial^{k+1}}{\partial t^{k+1}}\Gamma^{\alpha}_{A\bar\beta}=0$$

at $t=0$. Since $\Gamma^C_{AB}=\Gamma^C_{BA}$ and $\overline{\Gamma^C_{AB}}=\Gamma^{\bar C}_{\bar A\bar B}$, it is easy to see that $H_4(k+1)$ holds.

Next,

$$\begin{aligned}R_{AB\gamma\delta;E}=&R_{AB\gamma\delta,E}-\Gamma^G_{EA}R_{GB\gamma\delta}-\Gamma^G_{EB}R_{AG\gamma\delta}\\&-\Gamma^G_{E\gamma}R_{ABG\delta}-\Gamma^G_{E\delta}R_{AB\gamma G}.\end{aligned}$$

By $H_1(k+1)$, we have

$$\frac{\partial^{k+1}}{\partial t^{k+1}}R_{AB\gamma\delta,E}=(\frac{\partial^{k+1}}{\partial t^{k+1}}R_{AB\gamma\delta})_{,E}=0.$$

Since $H_1(i)$ and $H_4(i)$ are true for $0\le i\le k+1$, $H_5(k+1)$ is true. Since $H_1(i)$, $H_4(i)$ and $H_5(i)$ are true for $0\le i\le k+1$, $H_6(k+1)$ is true. Finally $H_6(i)$ is true for $0\le i\le k+1$ implies that $H_7(k+1)$ holds. Therefore, $H(k+1)$ holds. $\square$

Now we use the Uhlenbeck's trick to simplify the evolution equation of the complex curvature tensor. We pick an abstract vector bundle V over M which is isomorphic to $T_{\mathbb{C}}M$ and denote the isomorphism $u_0:V\to T_{\mathbb{C}}M$. And we

take $\{e_A := u_0^{-1}(\frac{\partial}{\partial z^A})\}$ as a basis of V. We also consider a metric h on V by $h := u_0^* g_0$. We let u_0 evolute by

$$\begin{cases} \frac{\partial}{\partial t}u(t) = \mathrm{Ric} \circ u(t) \\ u(0) = u_0 \end{cases}$$

In local coordinate, we have

$$\begin{cases} \frac{\partial}{\partial t}u_B^A = g^{AC}R_{CD}u_B^D, \\ u_B^A(0) = \delta_B^A \end{cases}$$

Consider metric $h(t) := u^*(t)g(t)$ on V for each $t \in [0,T]$. It is easy to see that $\frac{\partial}{\partial t}h(t) \equiv 0$ for all t, so $h(t) \equiv h$ for all t. We use $u(t)$ to pull the curvature tensor on $T_{\mathbb{C}}M$ back to V:

$$\widetilde{Rm}(e_A, e_B, e_C, e_D) := R(u(e_A), u(e_B), u(e_C), u(e_D)).$$

In local coordinate, we have

$$\tilde{R}_{ABCD} = R_{EFGH}u_A^E u_B^F u_C^G u_D^H$$

on $M \times [0,T]$. One can also check

$$\overline{h_{AB}} = h_{\bar{A}\bar{B}},\ \overline{\tilde{R}_{ABCD}} = \tilde{R}_{\bar{A}\bar{B}\bar{C}\bar{D}}.$$

Define a connection on V in the following: For any smooth section ξ on V, $X \in T_{\mathbb{C}}M$,

$$D_X^t \xi = u^{-1}(\nabla_X^t(u(\xi))).$$

One can check $D^t h = 0$ and $D^t u = 0$. We define $\triangle$ acting on any tensor on V by

$$\triangle := g^{EF}D_E^t D_F^t.$$

Then by (3.2), the evolution equation of $\tilde{R}$ is:

(3.3)

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{R}_{ABCD} =& \triangle\tilde{R}_{ABCD} - 2h^{EF}h^{GH}R_{EABG}R_{FHCD} - 2h^{EF}h^{GH}\tilde{R}_{EAGD}\tilde{R}_{FBHC} \\ &+ 2h^{EF}h^{GH}\tilde{R}_{EAGC}\tilde{R}_{FBHD}\end{aligned}$$

where $h^{AB} = (h^{-1})^{AB}$.

Lemma 3.2. *With the above notations, we have*

$$\frac{\partial^k}{\partial t^k}\tilde{R}_{AB\gamma\delta} = 0$$

at $t = 0$ for all A, B and γ, δ.

Proof. Note that we have:

$$\frac{\partial^k}{\partial t^k}\widetilde{R}_{AB\gamma\delta} = \sum_{\substack{m+n+p+q+r=k \\ 0 \le m,n,p,q,r \le k}} (u_A^E)_m (u_B^F)_n (u_\gamma^G)_p (u_\delta^H)_q (R_{EFGH})_r.$$

By Lemma 3.1, in order to prove the lemma, it is sufficient to prove that $\frac{\partial^k}{\partial t^k}u^{\alpha}_{\bar\beta}=0$ and $\frac{\partial^k}{\partial t^k}u^{\bar\alpha}_{\beta}=0$ for all k for all α,β at $t=0$.

Recall that

$$\begin{cases} \frac{\partial}{\partial t}u^A_B = g^{AC}R_{CD}u^D_B, \\ u^A_B(0)=\delta^A_B. \end{cases}$$

Hence $u^{\alpha}_{\bar\beta}=0$ and $u^{\bar\alpha}_{\beta}=0$. By induction, Lemma 3.1, and the fact that $u^A_B(0)=\delta^A_B$, one can prove that show $\frac{\partial^k}{\partial t^k}u^{\alpha}_{\bar\beta}=0$ and $\frac{\partial^k}{\partial t^k}u^{\bar\alpha}_{\beta}=0$ for all k. This completes the proof of the lemma. □

Proof of Theorem 3.1. As in [21], define a smooth function φ on $M\times[0,T]$ by

$$\begin{aligned}\varphi(z,t) &= h^{\alpha\bar\xi}h^{\beta\bar\zeta}h^{\gamma\bar\sigma}h^{\delta\bar\eta}\tilde R_{\alpha\beta\gamma\delta}\tilde R_{\bar\xi\bar\zeta\bar\sigma\bar\eta}+h^{\bar\alpha\xi}h^{\bar\beta\zeta}h^{\gamma\bar\sigma}h^{\delta\bar\eta}\tilde R_{\bar\alpha\bar\beta\gamma\delta}\tilde R_{\xi\zeta\bar\sigma\bar\eta} \\ &\quad + h^{\bar\alpha\xi}h^{\beta\bar\zeta}h^{\gamma\bar\sigma}h^{\delta\bar\eta}\tilde R_{\bar\alpha\beta\gamma\delta}\tilde R_{\xi\bar\zeta\bar\sigma\bar\eta}+h^{\alpha\bar\xi}h^{\bar\beta\zeta}h^{\gamma\bar\sigma}h^{\delta\bar\eta}\tilde R_{\alpha\bar\beta\gamma\delta}\tilde R_{\bar\xi\zeta\bar\sigma\bar\eta}\end{aligned} \tag{3.4}$$

One can check φ is well-defined (independent of coordinate changes on M) and is nonnegative. The evolution equation of φ is (See [21]):

$$(\frac{\partial}{\partial t}-\triangle)\varphi = \tilde R_{CDEF}*\tilde R_{GH\alpha\beta}*\tilde R_{AB\gamma\delta}-2g^{EF}\tilde R_{AB\gamma\delta;E}\overline{\tilde R_{AB\gamma\delta}}_{;F}. \tag{3.5}$$

As the real case, define the norm of the complex curvature tensor by:

$$|R_{ABCD}(t)|^2_{g(t)} = g^{AE}g^{BF}g^{CG}g^{DH}R_{ABCD}R_{EFGH}.$$

Then we have

$$|R_{ABCD}(t)| = |R_{ijkl}(t)| \le \frac{a}{t}$$

on $M\times[0,T]$ by assumption. By the definition of $\widetilde R_{ABCD}$, we also have:

$$|\widetilde R_{ABCD}(t)| = |R_{ABCD}(t)| \le \frac{a}{t}.$$

Combining with (3.5), we have

$$(\frac{\partial}{\partial t}-\triangle)\varphi \le \frac{C_1}{t}\varphi$$

on $M\times[0,T]$ for some constant C_1. Moreover,

$$\varphi \le |\widetilde R_{ABCD}(t)|^2 \le a^2/t^2.$$

On the other hand, by (3.4), Lemma 3.2 and the fact that h is independent of t, we conclude that at $t=0$,

$$\frac{\partial^k}{\partial t^k}\varphi = 0,$$

for all k. By Lemma 2.3, we conclude that $\varphi\equiv 0$ on $M\times[0,T]$. As in [21], we conclude that $g(t)$ is Kähler for all $t>0$.

□

Corollary 3.1. *Let (M^n, g_0) be a complete noncompact Kähler manifold with complex dimension n and with nonnegative complex sectional curvature. Suppose*

$$\inf_{p\in M}\{V_p(1)|\ p\in M\}=v_0>0.$$

where $V_p(1)$ is the volume of the geodesic ball with radius 1 and center at p with respect to g_0. Then there is $T>0$ depending only on n, v_0 such that the Kähler-Ricci flow has a complete solution on $M\times[0,T]$ such that $g(t)$ has nonnegative complex sectional curvature. Moreover the curvature satisfies:

$$|\mathrm{Rm}(g(t))|_{g(t)}\le\frac{c}{t}$$

where c is a constant depending only on n, v_0 with initial data $g(0)=g_0$.

Proof. The corollary follows immediately from the result of Cabezas-Rivas and Wilking[2], and Theorem 3.1. □

Corollary 3.2. *Let (M^n, g_0) be a complete noncompact Kähler manifold with complex dimension n and with nonnegative complex sectional curvature. Suppose M^n has maximum volume growth. Then M^n is biholomorphic to $\mathbb{C}^n$.*

Proof. By volume comparison, we have

$$\inf_{p\in M}\{V_p(1)|\ p\in M\}=v_0>0.$$

Let $g(t)$ be the solution of Kähler-Ricci flow on $M\times[0,T]$ obtained as in Corollary 3.1. Then for all $t>0$, $g(t)$ has nonnegative complex sectional curvature and the curvature of $g(t)$ is bounded. We want to prove that $g(t)$ has maximum volume growth.

Let $p\in M$ and let $r>0$ be fixed. Let $\tilde g(s)=r^{-2}g(r^2s)$, $0\le s\le r^{-2}T$. Then $\tilde g(s)$ is a solution to the Kähler-Ricci flow with initial data $\tilde g(0)=r^{-2}g_0$. Since the sectional curvature of $\tilde g(s)$ is nonnegative, as in [2], using a result of [17], one can prove that:

$$V_p(\tilde g(s),1)-V_p(r^{-2}g_0,1)=V_p(\tilde g(s),1)-V_p(\tilde g(0),1)\ge -c_ns$$

where $V_p(h,1)$ denotes the volume of the geodesic ball with radius 1 and center at p with respect to h and c_n is a positive constant depending only on n. Now

$$V_p(r^{-2}g_0,1)=\frac{V_p(g_0,r)}{r^{2n}}\ge v_0>0$$

because g_0 has maximum volume growth. Hence there is $r_0>0$ such that if $r\ge r_0$, then

$$r^{-2n}V_p(g(r^2s),r)=V_p(\tilde g(s),1)\ge C_1$$

for some constant independent of s and r for all $0\le s\le r^{-2}T$. Fix $t_0>0$, and let s be such that $r^2s=t_0$. Then $s\le r^{-2}T$. So we have

$$r^{-2n}V_p(g(t_0),r)\ge C_1$$

if r is large enough. That is, $g(t_0)$ has maximum volume growth. By [5], we conclude that M is biholomorphic to $\mathbb{C}^n$.

□

4. Preservation of non-negativity of holomorphic bisectional curvature

Let (M^n, g_0) be a complete noncompact Kähler manifold with complex dimension n. We want to study the preservation of non-negativity of holomorphic bisectional curvature under Kähler-Ricci flow, without assuming the curvature is bounded in space and time.

Let us first define a quadratic form for any $(0,4)$-tensor T on $T_{\mathbb{C}}M$ with a metric g by

$$\begin{aligned} Q(T)(X,\bar{X},Y,\bar{Y}) := &\sum_{\mu,\nu=1}^{n}(|T_{X\bar{\mu}\nu\bar{Y}}|^2 - |T_{X\bar{\mu}Y\bar{\nu}}|^2 + T_{X\bar{X}\nu\bar{\mu}}T_{\mu\bar{\nu}Y\bar{Y}}) \\ &-\sum_{\mu=1}^{n}\mathbf{Re}(T_{X\bar{\mu}}T_{\mu\bar{X}Y\bar{Y}} + T_{Y\bar{\mu}}T_{X\bar{X}\mu\bar{Y}}) \end{aligned}$$

for all $X, Y \in T^{1,0}_{\mathbb{C}}M$, where $T_{\alpha\bar{\beta}\gamma\bar{\delta}} = T(e_\alpha, \bar{e}_\beta, e_\gamma, \bar{e}_\delta)$, $T_{\alpha\bar{\beta}} = g^{\gamma\bar{\delta}}T_{\alpha\bar{\beta}\gamma\bar{\delta}}$ and $\{e_1, \ldots, e_n\}$ is a unitary frame with respect to the metric of g, $T_{X\bar{\mu}\nu\bar{Y}} = T(X, \bar{e}_\mu, e_\nu, \bar{Y})$ etc. Here T is a tensor has the following properties:

$$\overline{T(X,Y,Z,W)} = T(\bar{X},\bar{Y},\bar{Z},\bar{W});$$

$$T(X,Y,Z,W) = T(Z,W,X,Y) = T(X,W,Z,Y) = T(Y,X,W,Z).$$

Let $g(t)$ be a solution of the Kähler-Ricci flow:

$$\frac{\partial}{\partial t}g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}}.$$

Recall the evolution equation for holomorphic bisectional curvature: (See [8, Corollary 2.82])

$$\left(\frac{\partial}{\partial t} - \triangle\right)R(X,\bar{X},Y,\bar{Y}) = Q(R)(X,\bar{X},Y,\bar{Y})$$

for all $X, Y \in T^{1,0}_{\mathbb{C}}M$. Here $\triangle$ is with respect to $g(t)$.

Next define a $(0,4)$-tensor B on $T_{\mathbb{C}}M$ (with a metric g) by:

$$B(E,F,G,H) = g(E,F)g(G,H) + g(E,H)g(F,G)$$

for all $E, F, G, H \in T_{\mathbb{C}}M$.

Lemma 4.1. *In the above notation, $Q(B)(X,\bar{X},Y,\bar{Y}) \le 0$ for all $X, Y \in T^{1,0}_{\mathbb{C}}M$.*

Proof.

$$\begin{aligned} Q(B)(X,\bar{X},Y,\bar{Y}) = & \sum_{\mu,\nu=1}^{n}(|B_{X\bar{\mu}\nu\bar{Y}}|^2 - |B_{X\bar{\mu}Y\bar{\nu}}|^2 + B_{X\bar{X}\nu\bar{\mu}}B_{\mu\bar{\nu}Y\bar{Y}}) \\ & - \sum_{\mu=1}^{n}\mathbf{Re}(B_{X\bar{\mu}}B_{\mu\bar{X}Y\bar{Y}} + B_{Y\bar{\mu}}B_{X\bar{X}\mu\bar{Y}}) \end{aligned}$$

Let $\{e_1,\dots,e_n\}$ be a unitary frame. $X=\sum_{\mu=1}^n X^\mu e_\mu$, $\sum_{\mu=1}^n Y = Y^\mu e_\mu$.

We compute it term by term:

$$\begin{aligned} \sum_{\mu,\nu=1}^{n}|B_{X\bar{\mu}\nu\bar{Y}}|^2 = & \sum_{\mu,\nu=1}^{n}(g_{X\bar{\mu}}g_{\nu\bar{Y}} + g_{X\bar{Y}}g_{\bar{\mu}\nu})\cdot(g_{\bar{X}\mu}g_{\bar{\nu}Y} + g_{\bar{X}Y}g_{\mu\bar{\nu}}) \\ = & \sum_{\mu,\nu=1}^{n}(X^\mu\bar{Y}^\nu + g_{X\bar{Y}}g_{\bar{\mu}\nu})\cdot(\bar{X}^\mu Y^\nu + g_{\bar{X}Y}g_{\mu\bar{\nu}}) \\ = & |X|^2|Y|^2 + (n+2)|g(\bar{X},Y)|^2. \end{aligned}$$

$$\begin{aligned} \sum_{\mu,\nu=1}^{n}|B_{X\bar{\mu}Y\bar{\nu}}|^2 = & \sum_{\mu,\nu=1}^{n}(g_{X\bar{\mu}}g_{\bar{\nu}Y} + g_{X\bar{\nu}}g_{\bar{\mu}Y})\cdot(g_{\bar{X}\mu}g_{\nu\bar{Y}} + g_{\bar{X}\nu}g_{\mu\bar{Y}}) \\ = & \sum_{\mu,\nu=1}^{n}(X^\mu Y^\nu + X^\nu Y^\nu)\cdot(\bar{X}^\mu\bar{Y}^\nu + \bar{X}^\nu\bar{Y}^\mu) \\ = & 2|X|^2|Y|^2 + 2|g(\bar{X},Y)|^2. \end{aligned}$$

$$\begin{aligned} \sum_{\mu,\nu=1}^{n}B_{X\bar{X}\nu\bar{\mu}}B_{\mu\bar{\nu}Y\bar{Y}} = & (g_{X\bar{X}}g_{\nu\bar{\mu}} + g_{X\bar{\nu}}g_{\bar{X}\nu})\cdot(g_{\mu\bar{\nu}}g_{Y\bar{Y}} + g_{\mu\bar{Y}}g_{\bar{\nu}Y}) \\ = & (|X|^2 g_{\nu\bar{\mu}} + X^\mu\bar{X}^\nu)\cdot(g_{\mu\bar{\nu}}|Y|^2 + \bar{Y}^\mu Y^\nu) \\ = & (n+2)|X|^2|Y|^2 + |g(\bar{X},Y)|^2. \end{aligned}$$

$$\begin{aligned} \sum_{\mu=1}^{n}B_{X\bar{\mu}}B_{\mu\bar{X}Y\bar{Y}} = & \sum_{\mu=1}^{n}g^{k\bar{l}}B_{X\bar{\mu}k\bar{l}}B_{\mu\bar{X}Y\bar{Y}} \\ = & \sum_{\mu,\nu=1}^{n}B_{X\bar{\mu}\nu\bar{\nu}}B_{\mu\bar{X}Y\bar{Y}} \\ = & \sum_{\mu,\nu=1}^{n}(g_{X\bar{\mu}}g_{\nu\bar{\nu}} + g_{X\bar{\nu}}g_{\bar{\mu}\nu})\cdot(g_{\bar{X}\mu}g_{\bar{Y}Y} + g_{\bar{X}Y}g_{\mu\bar{Y}}) \\ = & \sum_{\mu,\nu=1}^{n}(X^\mu g_{\nu\bar{\nu}} + X^\nu g_{\bar{\mu}\nu})\cdot(\bar{X}^\mu g_{Y\bar{Y}} + \bar{Y}^\mu g_{\bar{X}Y}) \\ = & (n+1)|X|^2|Y|^2 + (n+1)|g(\bar{X},Y)|^2. \end{aligned}$$

Similarly, we have

$$\sum_{\mu=1}^{n} B_{Y\bar{\mu}} B_{X\bar{X}\mu\bar{Y}} = (n+1)|X|^2|Y|^2 + (n+1)|g(\bar{X}, Y)|^2$$

Therefore,

$$Q(B)(X, \bar{X}, Y, \bar{Y}) = -(n+1)(|X|^2|Y|^2 + |g(\bar{X}, Y)|^2) \leq 0.$$

□

We are ready to prove Theorem 1.2:

Theorem 4.1. *There is $0 < a(n) < 1$ depending only on n such that if $g(t)$ is a complete solution of Kähler-Ricci flow on $M \times [0, T]$ with $|\mathrm{Rm}(g(t))|_{g(t)} \leq \frac{a}{t}$, where M is an n-dimensional non-compact complex manifold. If $g(0)$ has nonnegative holomorphic bisectional curvature, then so does $g(t)$ for all $t \in [0, T]$.*

Proof. The theorem is known to be true if the curvature is uniformly bounded on space and time [21]. Since $g(t)$ has bounded curvature on $M \times [\tau, T]$ for all $\tau > 0$, it is sufficient to prove that $g(t)$ has nonnegative bisectional curvature on $M \times [0, \tau]$ for some $\tau > 0$. Hence we may assume that $T \leq 1$.

In the following, lower case $c_1, c_2, \cdots$ will denote constants depending only on n.

Since $g(T)$ has bound curvature $\frac{a}{T}$ and is complete, as in Lemma 2.2, a smooth function ρ defined on M such that

$$(1 + d_T(x, p)) \leq \rho(x) \leq D_1(1 + d_T(x, p))$$

$$|\bar{\nabla}\rho| + |\bar{\nabla}^2\rho| \leq D_1, \tag{4.1}$$

for some constant D_1 depending only on n and g_T, where $d_T(x, p)$ is the distance function with respect to g_T from a fixed point $p \in M$, where $\bar{\nabla}$ is the covariant derivative with respect to g_T

Suppose $|\mathrm{Rm}(g(t))|_{g(t)} \leq a/t$, where a is to be determined later depending only on n. By Lemma 2.2, we have

$$|\nabla\rho| \leq D_2 t^{-c_1 a} \tag{4.2}$$

for some constant D_2 depending only on n, g_T. Here and below, ∇ is the covariant derivative of $g(t)$ and hence is time dependent. We may get a better estimate for $\Delta\rho = \Delta_{g(t)}\rho$ than that in Lemma 2.2. Choose a normal coordinate with respect to $g(T)$ which also diagonalizes $g(t)$ with eigenvalues λ_α. Then

$$|\triangle\rho| = |g^{\alpha\bar{\beta}}\rho_{\alpha\bar{\beta}}| = \sum_{\alpha=1}^{n} \lambda_\alpha^{-1} |\bar{\nabla}^2\rho| \leq D_3 t^{-c_2 a}, \tag{4.3}$$

by Lemma 2.1.

Let ϕ be a smooth cut-off function from $\mathbb{R}$ to $[0,1]$ such that

$$\phi(x) = \begin{cases} 1, & x \le 1 \\ 0, & x \ge 2 \end{cases}$$

and $|\phi'| + |\phi''| \le D'$, $\phi' \le 0$. Let $\Phi = \phi^m$, where $m > 4$ is an integer to be determined later. Then

$$0 \ge \Phi' \ge -D(m)\Phi^q; \quad |\Phi''| \le D(m)\Phi^q$$

for some positive constant $D(m)$ depending only on D' and m, where $q = 1 - \frac{2}{m}$.

Let $\Psi(x) = \Phi(\frac{\rho(x)}{r})$ on M for $r \ge 1$. Note that Ψ depends on r.

Then we have

$$\begin{aligned} |\nabla\Psi| \le& \frac{1}{r} D(m)\Psi^q|\nabla\rho| \\ \le& \frac{D_4}{r}\Psi^q t^{-c_1 a} \end{aligned} \tag{4.4}$$

by (4.2), and

$$|\triangle\Psi| \le \frac{1}{r^2}|\Phi''||\nabla\rho|^2 + \frac{1}{r}|\Phi'\triangle\rho| \le \frac{D_4}{r}\Psi^q t^{-c_2 a} \tag{4.5}$$

by (4.3), where D_4 is a constant depending only on n, g_T, m.

For any $\varepsilon > 0$, we define a tensor A on $M \times (0,T]$: For vectors $X, Y, Z, W \in T_{\mathbb{C}}(M)$,

$$A(X,Y,Z,W) = t^{-\frac{1}{2}}\Psi(x)R(X,Y,Z,W) + \varepsilon B(X,Y,Z,W)$$

where R is the curvature tensor of $g(t)$ and B is evaluated with respect to $g(t)$.

Define the following function on $M \times (0,T]$:

$$H(x,t) = \inf\{A_{X\bar{X}Y\bar{Y}}(x,t) | |X|_t = |Y|_t = 1, X, Y \in T_x^{(1,0)}M\}.$$

Here $|\cdot|_t$ is the norm with respect to $g(t)$.

To show the theorem, it suffices to show for all $r >> 1$, $H(x,t) \ge 0$ for all x and for all $t > 0$. Note that $t^{\frac{1}{2}}H(x,t)$ is a continuous function. Since Ψ has compact support, and $B(X,\bar{X},Y,\bar{Y}) \ge 1$ for all $|X|_t = |Y|_t = 1$, there is a compact set $K \in M$ such that

$$H(x,t) > 0$$

on $(M \setminus K) \times (0,T]$. On the other hand, we claim that there is $T_0 > 0$ such that

$$t^{\frac{1}{2}}H(x,t) > 0$$

on $K \times (0,T_0)$. Let $\{e_1, e_2, \ldots, e_n\}$ be a unitary frame near a compact neighborhood U of a point $x_0 \in K$ with respect to g_0. Then at each point $x \in U$,

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}(x,t) = R_{\alpha\bar{\beta}\gamma\bar{\delta}}(x,0) + tE$$

where $|E|$ is uniformly bounded on $U\times[0,T]$. Since $g(t)$ is uniformly equivalent to $g(0)$ on U, for any $X,Y\in T^{1,0}_x(M)$ for $(x,t)\in U\times[0,T]$,

$$R(X,\bar X,Y,\bar Y)\ge -D_5t|X|_0^2|Y|_0^2$$

for some constant $D_5>0$ where we have used the fact that g_0 has nonnegative holomorphic bisectional curvature. Since $g(t)$ and g_0 are uniformly equivalent in K, and K is compact, we conclude that

$$R(X,\bar X,Y,\bar Y)\ge -D_6t$$

on $K\times[0,T]$ for some constant D_6 for all $X,Y\in T^{1,0}_x(M)$ with $|X|_t=|Y|_t=1$. Since $t^{\frac12}B(X,\bar X,Y,\bar Y)\ge t^{\frac12}$, it is easy to see the claim is true. To summarize, we have proved that there is a compact set K and there is $T_0>0$, such that $H(x,t)>0$ on $M\setminus K\times(0,T]$ and $K\times(0,T_0)$.

Suppose $H(x,t)<0$ for some $t>0$. Then $t^{\frac12}H(x,t)<0$ for some $t>0$. We must have $x\in K$ and $t\ge T_0$. Hence we can find $x_0\in K$, $t_0\ge T_0$ and a neighborhood V of x_0 such that $H(x_0,t_0)=0$, $H(x,t)\ge0$ for $x\in V$, $t\le t_0$. This implies that there exist $X_0,Y_0\in T^{(1,0)}_{x_0}M$ with norm $|X_0|_{g(t_0)}=|Y_0|_{g(t_0)}=1$ such that

$$A_{X_0\bar X_0Y_0\bar Y_0}(x_0,t_0)=0.$$

Then we extend X_0,Y_0 near x_0 by parallel translation with respect to $g(t_0)$ to vector fields $\widetilde X_0,\widetilde Y_0$ such that they are independent of time and

$$\Delta_{g(t_0)}\widetilde X_0=\Delta_{g(t_0)}\widetilde Y_0=0,$$

at x_0.

Denote $h(x,t):=A_{\widetilde X_0\bar{\widetilde X}_0\widetilde Y_0\bar{\widetilde Y}_0}(x,t)$. At (x_0,t_0), we have $h(x_0,t_0)=0$ and $h(x,t)\ge0$ for $x\in V$, $t\le t_0$ by the definition of A.

Hence at (x_0,t_0),

(4.6)
$$\begin{aligned}
0\ge&\left(\frac{\partial}{\partial t}-\triangle\right)h\\
=&t_0^{-\frac12}\Psi\left(\left(\frac{\partial}{\partial t}-\Delta\right)R\right)(X_0,\bar X_0,Y_0,\bar Y_0))-t_0^{-\frac12}R(X_0,\bar X_0,Y_0,\bar Y_0)\Delta\Psi\\
&-2t_0^{-\frac12}\langle\nabla R(\widetilde X_0,\bar{\widetilde X}_0,\widetilde Y_0,\bar{\widetilde Y}_0),\nabla\Psi\rangle-\frac12t_0^{-\frac32}\Psi R(X_0,\bar X_0,Y_0,\bar Y_0)-\varepsilon(\Delta B)(X_0,\bar X_0,Y_0,\bar Y_0)\\
&+\varepsilon(-\mathrm{Ric}(X_0,\bar X_0)-\mathrm{Ric}(Y_0,\bar Y_0)-\mathrm{Ric}(X_0,\bar Y_0)g(X_0,\bar Y_0)-\mathrm{Ric}(\bar X_0,Y_0)g(X_0,\bar Y_0))\\
\ge&t_0^{-\frac12}\Psi Q(R)(X_0,\bar X_0,Y_0,\bar Y_0))-D_4r^{-1}t_0^{-\frac12-c_2a}\Psi^q|R(X_0,\bar X_0,Y_0,\bar Y_0))|\\
&-\frac12t_0^{-\frac32}\Psi R(X_0,\bar X_0,Y_0,\bar Y_0)-c_3\varepsilon at_0^{-1}-2t_0^{-\frac12}\langle\nabla R(\widetilde X_0,\bar{\widetilde X}_0,\widetilde Y_0,\bar{\widetilde Y}_0),\nabla\Psi\rangle,
\end{aligned}$$

where we have used (4.5) and the fact that $\Delta B = 0$. On the other hand, at (x_0, t_0)

$$
\begin{aligned}
0 =& \nabla h \\
=& t_0^{-\frac{1}{2}} \nabla \left(R(\widetilde{X}_0, \bar{\widetilde{X}}_0, \widetilde{Y}_0, \bar{\widetilde{Y}}_0)\Psi \right) \\
=& t_0^{-\frac{1}{2}} \left[\Psi \nabla R(\widetilde{X}_0, \bar{\widetilde{X}}_0, \widetilde{Y}_0, \bar{\widetilde{Y}}_0) + R(X_0, \bar{X}_0, Y_0, \bar{Y}_0) \nabla \Psi \right]
\end{aligned}
$$

where we have used the fact that $\nabla g = 0$ and $\nabla \widetilde{X}_0 = \nabla \widetilde{Y}_0 = 0$ at (x_0, t_0). Hence (4.6) implies

(4.7)

$$
\begin{aligned}
0 \geq & t_0^{-\frac{1}{2}} \Psi Q(R)(X_0, \bar{X}_0, Y_0, \bar{Y}_0)) - D_7 r^{-1} t_0^{-\frac{1}{2} - c_4 a} |R(X_0, \bar{X}_0, Y_0, \bar{Y}_0)| \left(\Psi^q + \Psi^{2q-1} \right) \\
& - \frac{1}{2} t_0^{-\frac{3}{2}} \Psi R(X_0, \bar{X}_0, Y_0, \bar{Y}_0) - c_3 \varepsilon a t_0^{-1}
\end{aligned}
$$

where we have used (4.4), where $D_7 > 0$ is a constant depending only on g_T, n, m. On the other hand, by the null-vector condition [15, Proposition 1.1] (see also [1]), we have

$$
Q(A)(X_0, \bar{X}_0, Y_0, \bar{Y}_0) \geq 0
$$

By a direct computation, one can see that

$$
Q(A) = t_0^{-1} \Psi^2 Q(R) + \varepsilon^2 Q(B) + t_0^{-\frac{1}{2}} \Psi \varepsilon R * B,
$$

and we have

$$
\begin{aligned}
0 \leq & t_0^{-1} \Psi^2 Q(R)(X_0, \bar{X}_0, Y_0, \bar{Y}_0) + \varepsilon^2 Q(B)(X_0, \bar{X}_0, Y_0, \bar{Y}_0) + c_5 \varepsilon \Psi a t_0^{-\frac{3}{2}} \\
\leq & t_0^{-1} \Psi^2 Q(R)(X_0, \bar{X}_0, Y_0, \bar{Y}_0) + c_5 \varepsilon \Psi a t_0^{-\frac{3}{2}}
\end{aligned}
$$

where we have used Lemma 4.1 and c_5 is a constant depending only on n. That is

$$
\Psi Q(R)(X_0, \bar{X}_0, Y_0, \bar{Y}_0) \geq -c_5 \varepsilon a t_0^{-\frac{1}{2}} \tag{4.8}
$$

where we have used the fact that $h(x_0, t_0) = 0$ which implies $\Psi(x_0, t_0) > 0$.

Combining this with (4.7), we have

$$
\begin{aligned}
0 \geq & -(c_3 + c_5) \varepsilon a t_0^{-1} - D_7 r^{-1} t_0^{-\frac{1}{2} - c_4 a} |R(X_0, \bar{X}_0, Y_0, \bar{Y}_0)| \left(\Psi^q + \Psi^{2q-1} \right) \\
& - \frac{1}{2} t_0^{-\frac{3}{2}} \Psi R(X_0, \bar{X}_0, Y_0, \bar{Y}_0),
\end{aligned} \tag{4.9}
$$

Since $h(x_0, t_0) = 0$, we also have

$$
\Psi(x_0, t_0) R(X_0, \bar{X}_0, Y_0, \bar{Y}_0) = -t_0^{\frac{1}{2}} \varepsilon B(X_0, \bar{X}_0, Y_0, \bar{Y}_0).
$$

Hence at (x_0, t_0), (4.9) implies, if $0 < a < 1$, then

$$
\begin{aligned}
0 \geq & -(c_3+c_5)\varepsilon a - 2D_7 r^{-1} t_0^{\frac{1}{2}-c_4 a} |R(X_0,\bar{X}_0,Y_0,\bar{Y}_0)| \Psi^{2q-1} - \frac{1}{2} t_0^{-\frac{1}{2}} \Psi R(X_0,\bar{X}_0,Y_0,\bar{Y}_0) \\
\geq & -(c_3+c_5)\varepsilon a - 2D_7 r^{-1} t_0^{\frac{1}{2}-c_4 a} |R(X_0,\bar{X}_0,Y_0,\bar{Y}_0)|^{2(1-q)} \, |\varepsilon t_0^{\frac{1}{2}} B(X_0,\bar{X}_0,Y_0,\bar{Y}_0)|^{2q-1} \\
& + \frac{1}{2}\varepsilon B(X_0,\bar{X}_0,Y_0,\bar{Y}_0) \\
\geq & -(c_3+c_5)\varepsilon a - D_8 r^{-1} \varepsilon^{2q-1} t_0^{\alpha} + \frac{1}{2}\varepsilon
\end{aligned}
$$

because $0 \leq \Psi \leq 1$, $q = 1 - \frac{2}{m} < 1$, $m > 4$ where $D_8 > 0$ are constants depending only on g_T, n, m. Here

$$
\alpha = \frac{1}{2} - c_4 a - 2(1-q) + \frac{1}{2}(2q-1) = 3q - c_4 a - 2.
$$

Hence if $c_4 a < \frac{1}{2}$ and $a < 1$, then a depends only on n and $3q - c_4 a - 2 > 0$, provided m is large enough. If a, m are chosen satisfying these conditions, then we have

$$
0 \geq -(c_3+c_5)\varepsilon a - D_8 r^{-1} \varepsilon^{2q-1} + \frac{1}{2}\varepsilon.
$$

If a also satisfies $a(c_3+c_5) < \frac{1}{2}$, then we have a contradiction if r is large enough. Hence if

$$
0 < a < \min\{1, \frac{1}{2}c_4^{-1}, \frac{1}{2}(c_3+c_5)^{-1}\},
$$

then $g(t)$ will have nonnegative holomorphic bisectional curvature. This completes the proof of the theorem.

□

As an application, we have the following:

Corollary 4.1. *Let (M^n, g_0) is a complete noncompact Kähler manifold with complex dimension $n \geq 2$ with nonnegative holomorphic bisectional curvature with maximum volume growth. Suppose there is $r_0 > 0$ and there is $C > 0$ such that*

$$
\left(\fint_{B_x(r_0)} |\mathrm{Rm}|^p \right)^{\frac{1}{p}} \leq C
$$

for some $p > n$ for all $x \in M$. Then M is biholomorphic to $\mathbb{C}^n$.

Proof. By [23], the Ricci flow with initial data g_0 has short time solution $g(t)$ so that the curvature has the following bound:

$$
|\mathrm{Rm}(g(t))| \leq C t^{-\frac{n}{p}}
$$

for some constant C. Since $\frac{n}{p} < 1$, by Theorems 3.1and 4.1 $g(t)$ is Kähler and has bounded nonnegative bisectional curvature for $t > 0$. Since $\frac{n}{p} < 1$ it is easy to see that $g(t)$ is uniformly equivalent to g_0. Hence $g(t)$ also has maximum volume growth. By [5], M is biholomorphic to $\mathbb{C}^n$.

□

5. PRODUCING KÄHLER-RICCI FLOW THROUGH h-FLOW

We want to produce solutions to Kähler-Ricci flow using the solutions of the so-called h-flow by M. Simon [18]. Let us recall the set up and some results in [18]. Let M^n be a smooth manifold, and let g and h be two Riemannian metrics on M. For a constant $\delta > 1$, h is said to be δ close to g if

$$\delta^{-1}h \le g \le \delta h.$$

Let $g(t)$ be a smooth family of metrics on $M \times [0,T]$, $T > 0$. $g(t)$ is said to be a solution to the h-flow, if $g(t)$ satisfies following DeTurck flow, see [20, 18]:

$$\frac{\partial}{\partial t} g_{ij} = -2\mathrm{Ric}_{ij} + \nabla_i V_j + \nabla_j V_i, \tag{5.1}$$

where

$$V_i = g_{ij} g^{kl} (\Gamma^j_{kl} - {}^h\Gamma^j_{kl}),$$

and Γ^i_{kl}, ${}^h\Gamma^i_{kl}$ are the Christoffel symbols of $g(t)$ and h respectively, and ∇ is the covariant derivative with respect to $g(t)$. One can rewrite (5.1) in the following way which shows that it is a strictly parabolic system:

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} =& g^{\alpha\beta}\widetilde{\nabla}_\alpha \widetilde{\nabla}_\beta g_{ij} - g^{\alpha\beta} g_{ip} h^{pq} \widetilde{\mathrm{Rm}}_{j\alpha q\beta} - g^{\alpha\beta} g_{jp} h^{pq} \widetilde{\mathrm{Rm}}_{i\alpha q\beta} \\ &+ \frac{1}{2} g^{\alpha\beta} g^{pq} (\widetilde{\nabla}_i g_{p\alpha} \cdot \widetilde{\nabla}_j g_{q\beta} + 2\widetilde{\nabla}_\alpha g_{jp} \cdot \widetilde{\nabla}_q g_{i\beta} - 2\widetilde{\nabla}_\alpha g_{jp} \cdot \widetilde{\nabla}_\beta g_{iq} \\ &- 2\widetilde{\nabla}_j g_{\alpha p} \cdot \widetilde{\nabla}_\beta g_{iq} - 2\widetilde{\nabla}_i g_{\alpha p} \cdot \widetilde{\nabla}_\beta g_{jq}),\end{aligned}$$

where $\widetilde{\nabla}$ is covariant derivative with respect to h.

In order to emphasis the background metric h, we call it h-flow as in [18]. We are only interested in the case that M is noncompact and g is complete.

In [18], Simon obtained the following:

Theorem 5.1. *[Simon] There is a $\epsilon = \epsilon(n) > 0$ depending only on n such that if (M^n, g_0) is a smooth n-dimensional complete noncompact manifold such that there is a smooth Riemannian metric h with $|\nabla^i \mathrm{Rm}(h)| \le k_i$ for all i and is $(1+\epsilon(n))$ close to g_0, then the h-flow (5.1) has a smooth solution on $M \times [0,T]$ for some $T > 0$ with T depending only on n, k_0 such that $g(t) \to g_0$ as $t \to 0$ uniformly on compact sets and such that*

$$\sup_{x\in M} |\nabla^i g(t)|^2 \le \frac{C_i}{t^i}$$

for all i, where C_i depends only on $n, k_0, \dots, k_i$. Moreover, h is $(1+2\epsilon)$ close to $g(t)$ for all t. Here and in the following ∇ and $|\cdot|$ are with respect to h.

From this and using Theorem 3.1, one can construct solution $g(t)$ to the Kähler-Ricci flow if g_0 is Kähler . Moreover, the curvature of $g(t)$ is bounded by C/t. However, we motivated by the uniformization conjecture of Yau [25], we also want to prove that if g_0 has nonnegative holomorphic bisectional curvature, then one can construct solution to the Kähler-Ricci flow so that $g(t)$

also has nonnegative holomorphic bisectional curvature. To achieve this goal, we want to apply Theorem 4.1. Therefore, we want to show that C in the curvature bound C/t above is small provided ϵ is small. We need more refined estimates of $|\nabla g|$ and $|\nabla^2 g|$. We proceed as in [20, 18]

Recall the evolution equations $|\nabla^p g|^2$, $p=1,2$. Let $\widetilde{\mathrm{Rm}}$ be the curvature tensor of h and let

$$\square = \frac{\partial}{\partial t} - g^{ij}\nabla_i\nabla_j.$$

Then (see [18]):

$$\begin{aligned}\square|\nabla g|^2 =& -2g^{kl}\nabla_k\nabla g_{ij}\cdot\nabla_l\nabla g_{ij}\\ &+\widetilde{\mathrm{Rm}}*g^{-1}*\nabla g*\nabla g+\widetilde{\mathrm{Rm}}*g^{-1}*g^{-1}*g*\nabla g*\nabla g\\ &+g^{-1}*g*\nabla\widetilde{\mathrm{Rm}}*\nabla g+g^{-1}*g^{-1}*\nabla g*\nabla g*\nabla^2 g\\ &+g^{-1}*g^{-1}*g^{-1}*\nabla g*\nabla g*\nabla g*\nabla g,\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\square(|\nabla^2 g|^2) =& -2g^{ij}\nabla_i(\nabla^2 g)\nabla_j(\nabla^2 g)\\ &+\sum_{i+j+k=2,0\le i,j,k\le 2}\nabla^i g^{-1}*\nabla^j g*\nabla^k\widetilde{\mathrm{Rm}}*\nabla^2 g\\ &+\sum_{i+j+k+l=4,0\le i,j,k,l\le 3}\nabla^i g^{-1}*\nabla^j g^{-1}*\nabla^k g*\nabla^l g*\nabla^2 g.\end{aligned} \tag{5.3}$$

Here for tensors $S_1 * S_2$ denotes some trace with respect to h of tensors S_1, S_2. The total numbers of terms on the R.H.S. of each equation depend only on n.

Lemma 5.1. *Let (M^n, h) be a complete noncompact Riemannian manifold such that $|\mathrm{Rm}(h)| \le k_0$, and $|\nabla\mathrm{Rm}(h)| \le k_1$ with $k_0 + k_1 \le 1$. For any $\alpha > 0$ there is a constant $b(n,\alpha) > 0$ depending only on n and α such that $e^{2b} \le 1+\epsilon(n)$ where $\epsilon(n)$ is the constant in Theorem 5.1, and if $g(t)$ is the solution of the h-flow on $M\times[0,T]$, $T\le 1$ obtained in Theorem 5.1 with $g(0) = g_0$ satisfies $e^{-b}h \le g_0 \le e^b h$, then there is a $T_1(n,\alpha) > 0$ depending only on n,α such that*

$$|\nabla g(t)|^2 \le \frac{\alpha}{t}$$

for all $t\in(0,T_1]$.

Proof. By a $\mathcal{C}^0$-estimate of h-flow (See [20, Theorem 2.5] or [18, Theorem 2.3]), the constant $\epsilon(n) > 0$ in Theorem 5.1 can be chosen such that if h is e^b close to g_0 with $e^b \le 1+\epsilon(n)$, then the solution $g(t)$ in Theorem 5.1 is defined on $M\times[0,T_2]$ for some $T \ge T_2 = T_2(n) > 0$ (note that we assume $k_0+k_1\le 1$. Moreover

$$e^{-2b}h \le g(t) \le e^{2b}h \tag{5.4}$$

for all $t \in [0, T_2]$.

Let $f_0 = |g|$, $f_1 = |\nabla g|$ and $f_2 = |\nabla^2 g|$. First choose $b > 0$ such that:

(**c1**) $e^{2b} \le 2$ and $e^b \le 1 + \epsilon(n)$.

Then we have $f_0 \le c_1$. Here and in the following, lower case c_i will denote positive constants depending only on n.

Using (5.4), we estimate terms of R.H.S. of (5.2) in the following:

$$\begin{aligned}
&g^{\alpha\beta}\nabla_\alpha\nabla g_{ij}\cdot\nabla_\beta\nabla g_{ij} \ge \frac{1}{2}f_2^2\\
&\widetilde{\mathrm{Rm}} * g^{-1} * \nabla g * \nabla g \le c_2 k_0 f_1^2\\
&\widetilde{\mathrm{Rm}} * g^{-1} * g^{-1} * g * \nabla g * \nabla g \le c_2 k_0 f_1^2\\
&g^{-1} * g * \nabla\widetilde{\mathrm{Rm}} * \nabla g \le c_2 k_1 f_1\\
&g^{-1} * g^{-1} * \nabla g * \nabla g * \nabla^2 g \le c_2 f_1^2 \cdot f_2\\
&g^{-1} * g^{-1} * g^{-1} * \nabla g * \nabla g * \nabla g * \nabla g \le c_2 f_1^4.
\end{aligned}$$

Then (5.2) implies

$$\begin{aligned}
\Box(f_1^2) \le & - f_2^2 + c_2\left(2k_0 f_1^2 + k_1 f_1 + f_1^2 f_2 + f_1^4\right)\\
\le & - \frac{1}{2}f_2^2 + c_3\left(f_1^4 + 1\right),
\end{aligned} \tag{5.5}$$

where we have used the assumption that $k_0 + k_1 \le 1$. Next we define a smooth function φ on $M \times [0, T]$ as follows:

$$\varphi = a(n) + g_{j_1 i_1} h^{i_1 j_2} g_{j_2 i_2} h^{i_2 j_3} \cdots g_{j_m i_m} h^{i_m j_1}.$$

We will choose $a > 0$ and m later with a depending only on n and m depending only on n, α. One can choose a coordinate system $\{x^i\}$ such that at one point: $h_{ij} = \delta_{ij}$ and $g_{ij} = \lambda_i \delta_{ij}$. Then $\varphi = a + \sum_{i=1}^{n} \lambda_i^m$. By direct computation, we have

$$\begin{aligned}
\Box\varphi = & m\lambda_k^{m-1} * (\widetilde{\mathrm{Rm}} * g^{-1} * g + g^{-1} * g^{-1} * \nabla g * \nabla g)\\
& - m(\lambda_i^{m-2} + \lambda^{m-3}\lambda_j + \cdots + \lambda_j^{m-2}) g^{\alpha\beta}\nabla_\alpha g_{ij}\nabla_\beta g_{ij}\\
\le & c_4 m e^{2b(m-1)}\left(f_1^2 + 1\right) - \frac{m(m-1)}{2} e^{-2b(m-2)} f_1^2.
\end{aligned} \tag{5.6}$$

Here we use the fact that $e^{-2b}h \le g(t) \le e^{2b}h$ and $k_0 \le 1$.

Now we define $\psi = \varphi \cdot f_1^2$. By (5.5) and (5.6), we have

(5.7)
$$
\begin{aligned}
\Box\psi =&\varphi \cdot \Box(f_1^2) + \Box\varphi \cdot f_1^2 - 2g^{ij}\nabla_i\varphi\nabla_j f_1^2\\
\le&\varphi\left(-\frac{1}{2}f_2^2 + c_3\left(f_1^4+1\right)\right) + f_1^2\left(c_4 m e^{2b(m-1)}\left(f_1^2+1\right) - \frac{m(m-1)}{2}e^{-2b(m-2)}f_1^2\right)\\
&+\frac{\varphi}{2}f_2^2 + \frac{c_5 m^2 e^{2b(m-1)}}{a}f_1^4\\
\le&\varphi\left(c_3\left(f_1^4+1\right)\right) + f_1^2\left(c_4 m e^{2b(m-1)}\left(f_1^2+1\right) - \frac{m(m-1)}{2}e^{-2b(m-2)}f_1^2\right)\\
&+\frac{c_5 m^2 e^{2b(m-1)}}{a}f_1^4,
\end{aligned}
$$

where we have used the fact (5.4) and $e^{2b} \le 2$ which also imply

$$
-2g^{ij}\nabla_i\varphi\nabla_j f_1^2 \le cm(\sum_{i=1}^n \lambda_i^{m-1})f_1^2 f_2
$$

for some constant c depending only on n. Now let b, m be such that

(**c2**) $b = \frac{1}{2m}$ and $m \ge 2$ with $e^{1/m} \le 2$.

Note that if $m \ge 2$, we have $m-1 \ge \frac{m}{2}$, and $e^{bm} = e^{1/2}$. Hence the above inequality becomes:

$$
\Box\psi \le c_7(a+1)\left(f_1^4+1\right) + c_7 m f_1^2\left(1+f_1^2\right) - c_8 m^2 f_1^4 + \frac{c_7 m^2}{a}f_1^4.
$$

Let $a = \frac{2c_7}{c_8}$, then $a = a(n)$ which depends only on n. We have

$$
\Box\psi \le c_9\left(m f_1^4 + m f_1^2 + 1\right) - \frac{1}{2}c_8 m^2 f_1^4. \tag{5.8}
$$

Since h has bounded curvature, there is a smooth function $\rho(x)$ such that

$$
d(p,x) + 1 \le \rho(x) \le D_1\left(d(p,x)+1\right), |\nabla\rho| + |\nabla^2\rho| \le D_1
$$

where $d(p,x)$ is the distance function from p with respect to h and D_1 is a constant depending on h, see [21, 22]. Let $\bar\eta(s)$ be a smooth function on $\mathbb{R}$ such that $0 \le \bar\eta \le 1$, $\bar\eta = 1$ for $s \le 1$, $\bar\eta = 0$ for $s \ge 2$, $|\bar\eta'|^2 \le D_2\bar\eta$ and $|\bar\eta''| \le D_2$. For any $r \ge 1$, let

$$
F(x,t) = t\eta_r(x)\psi(x) = t\eta_r(x)\varphi(x)f_1^2(x),
$$

where $\eta_r(x) = \bar\eta(\frac{\rho(x)}{r})$. By (5.8), we have

$$
\Box F \le t\eta_r\left[c_9\left(m f_1^4 + m f_1^2 + 1\right) - \frac{1}{2}c_8 m^2 f_1^4\right] + \eta_r\psi - t\psi\Box\eta_r - 2tg^{ij}\nabla_i\eta_r\nabla_j\psi.
$$

Suppose let $F(x_0,t_0) = \max_{(x,t)\in M\times[0,T]} F(x,t)$. Suppose $t_0 = 0$, then $F(x_0,t_0) = 0$. Suppose $t_0 > 0$, then at (x_0,t_0), $\psi\nabla_j\eta_r + \eta_r\nabla_j\psi = 0$, and multiplying the

about inequality by $t_0\eta_r$, we have

$$\begin{aligned}0 \leq &(t_0\eta_r)^2\left[c_9\left(mf_1^4+mf_1^2+1\right)-\frac{1}{2}c_8m^2f_1^4\right]+t_0\eta_r^2\psi+D_3r^{-1}t_0^2\eta_r\psi\\ \leq &c_{10}(mF^2+mF+1)-c_{11}m^2F^2+(1+D_3r^{-1})F\end{aligned}$$

where we have used the fact that $t_0\leq T\leq 1$, $\eta_r\leq 1$, and $c^{-1}\leq\varphi\leq c$ for some constant c depending only on n. Let m be such that

(**c3**) $m\geq\frac{2c_{10}}{c_{11}}$.

Then at (x_0,t_0)

$$0\leq -\frac{1}{2}c_{11}m^2F^2+\left(c_{10}m+1+D_3r^{-1}\right)F+c_{10}$$

and

$$F(x_0,t_0)\leq\frac{2\left(c_{10}m+1+D_3r^{-2}\right)+\left(2c_{10}c_{11}m^2\right)^{\frac{1}{2}}}{c_{11}m^2}.$$

Let $r\to\infty$, we conclude that:

$$\sup_{M\times[0,T]}t\varphi f_1^2\leq\frac{c_{12}}{m},$$

and so

$$\sup_{M\times[0,T]}t|\nabla g|^2\leq\frac{c_{13}}{m}\leq\alpha \tag{5.9}$$

provided

(**c4**) $m\geq\frac{c_{13}}{\alpha}$.

Hence if we choose m large enough so that m satisfies (**c3**), (**c4**) such that $e^{1/m}\leq 2$. Note that m depends only on n and α. Then choose $b=\frac{1}{2m}$ and satisfies (**c1**). b also satisfies (**c2**). Then b depends only on n,α. For this choice of m,b we conclude the lemma is true by (5.9).

□

Lemma 5.2. *Let (M^n,h) be a complete noncompact Riemannian manifold such that $|\mathrm{Rm}(h)|\leq k_0$, $|\nabla\mathrm{Rm}(h)|\leq k_1$, $|\nabla^2\mathrm{Rm}(h)|\leq k_2$ with $k_0+k_1+k_2\leq 1$. For any $\alpha>0$ there is a constant $b(n,\alpha)>0$ depending only on n and α such that $e^{2b}\leq 1+\epsilon(n)$ where $\epsilon(n)$ is the constant in Theorem 5.1, and if $g(t)$ is the solution of the h-flow on $M\times[0,T]$, $T\leq 1$ obtained in Theorem 5.1 with $g(0)=g_0$ satisfies $e^{-b}h\leq g_0\leq e^bh$, then there is a $T\geq T_2(n,\alpha)>0$ depending only on n,α such that*

$$|\nabla^2 g(t)|^2\leq\frac{\alpha^2}{t^2}$$

for all $t\in(0,T_2]$.

Proof. As in the proof of the previous lemma, let $f_i=|\nabla^i g|$. Let $b>0$ be such that $e^b\leq 1+\epsilon$ where $\epsilon=\epsilon(n)$ is the constant in Theorem 5.1. Suppose h is e^b close to g_0 and let $g(t)$ be the solution of the h-flow with $g(0)=g_0$ obtained

in Theorem 5.1. Let $\beta > 0$ to be chosen later depending only on n, α. By Lemma 5.1, we assume that:

(**c1**) $e^{2b} \le 2$ is such that there is $T \ge T_1(n,\beta) > 0$ and $f_1^2 \le \frac{\beta}{t}$ on $M\times[0,T_1]$. Note that b depends only on n and β. We may also assume that

$$e^{-2b}h \le g \le e^{2b}g.$$

as in the proof of the previous lemma. As before, in the following, lower case c_i will denote a positive constant depending only on n.

By (5.3), we have

$$\begin{aligned}\Box f_2^2 \le& - f_3^2 + c_1(f_2 + f_1f_2 + f_1^2f_2 + f_2^2 + f_1f_2f_3 + f_2^3 + f_1^2f_2^2 + f_1^4f_2)\\ \le& -\frac{1}{2}f_3^2 + c_2\left(f_2 + f_1f_2 + f_1^2f_2 + f_2^2 + f_2^3 + f_1^2f_2^2 + f_1^4f_2\right)\\ \le& -\frac{1}{2}f_3^2 + c_3\left(f_2^2 + f_2^3 + f_1^2f_2^2 + f_1^2 + f_1^4 + f_1^6 + 1\right)\\ \le& -\frac{1}{2}f_3^2 + c_4\left(f_2^3 + \frac{\beta}{t}f_2^2 + \left(\frac{\beta}{t}\right)^3 + 1\right)\end{aligned} \tag{5.10}$$

provided that

(**c2**) $t \le \beta$.

Here we have used that fact that $f_0 \le c$, $f_1 \le \frac{\beta}{t}$ and $k_0 + k_1 + k_2 \le 1$. In the following, we always assume that (**c2**) is true.

Let $\psi(x,t) = (at^{-1} + f_1^2)\, f_2^2$, where $a > 0$ is a constant depending only on n and β and will be chosen later.

Combine (5.5) and (5.10), we have

$$\begin{aligned}\Box\psi =&(at^{-1} + f_1^2)\Box f_2^2 + f_2^2\Box f_1^2 - 2g^{ij}\nabla_i(f_1^2)\cdot\nabla_j(f_2^2) - at^{-2}f_2^2\\ \le&(at^{-1} + f_1^2)\left(-\frac{1}{2}f_3^2 + c_4\left(f_2^3 + \frac{\beta}{t}f_2^2 + \left(\frac{\beta}{t}\right)^3 + 1\right)\right)\\ &+ f_2^2\left(-\frac{1}{2}f_2^2 + c_5\left(f_1^4 + 1\right)\right) + c_5f_1f_3f_2^2\\ \le&\left(c_6f_1^2 - \frac{1}{2}at^{-1}\right)f_3^2 + c_4(at^{-1} + f_1^2)\left(f_2^3 + \frac{\beta}{t}f_2^2 + \left(\frac{\beta}{t}\right)^3 + 1\right)\\ &+ f_2^2\left(-\frac{1}{4}f_2^2 + c_5\left(f_1^4 + 1\right)\right)\\ \le& -\frac{1}{8}f_2^4 + c_6\left(\frac{\beta}{t}\right)^4\end{aligned} \tag{5.11}$$

where we have chosen a so that

(**c3**) $a = 2c_6\beta$ which depends only on n and β.

Here we have used the fact that $t \le \beta$.

For $r > 1$, let ρ, $\bar{\eta}$, η_r as in the proof of the previous lemma, and let $F = t^p\eta_r\psi = t^p\eta_r(at^{-1} + f_1^2)f_2^2$, $p \geq 2$. Let

$$F(x_0, t_0) = \max_{(x,t)\in M\times[0,T_1]} F(x,t).$$

If $t_0 = 0$, then $F(x_0, t_0) = 0$. If $t_0 > 0$, then at (x_0, t_0), we have by (5.11), as in the proof of the previous lemma:

$$\begin{aligned}0 \leq & t_0^p\eta_r\left(-\frac{1}{8}f_2^4 + c_6\left(\frac{\beta}{t_0}\right)^4\right) + pt_0^{p-1}\eta_r\psi + t_0^p\psi\Box\eta_r - 2t_0^p\nabla_i\eta_r\nabla_j\psi\\ \leq & t_0^p\eta_r\left(-\frac{1}{8}f_2^4 + c_6\left(\frac{\beta}{t_0}\right)^4\right) + pt_0^{p-1}\eta_r\psi + D_1r^{-1}t_0^p\psi\end{aligned}$$

where D_1 depends on h. Multiply both sides by $t_0^p\eta_r(at_0^{-1} + f_1^2)^2$ using the fact that $t_0 \leq 1, \eta_r \leq 1$ and that $t_0 \leq \beta$, we have

$$\begin{aligned}\frac{1}{8}F^2 \leq & c_6t_0^{2p}\eta_r(at_0^{-1} + f_1^2)^2\left(\frac{\beta}{t_0}\right)^4 + pt_0^{2p-1}\eta_r^2(at_0^{-1} + f_1^2)^2\psi + D_1r^{-1}t_0^{2p}\eta_r(at_0^{-1} + f_1^2)^2\psi\\ \leq & c_7t_0^{2p}\left(\frac{\beta}{t_0}\right)^6 + c_8F\left(\frac{\beta}{t_0}\right)^2\left(pt_0^{p-1} + D_1r^{-1}t_0^p\right).\end{aligned}$$

Let $p = 3$, we have

$$\frac{1}{8}F^2 \leq c_9\left(\beta^6 + \beta^2F\left(1 + D_1r^{-1}\right)\right)$$

Hence

$$F(x_0, t_0) \leq c_{10}\left(\beta^3 + \beta^2(1 + D_1r^{-1})\right).$$

Let $r \to \infty$, we conclude that

$$\sup_{(x,t)\in M\times[0,T_2]} t^3(at^{-1} + f_1^2)f_2^2 \leq c_{10}\left(\beta^3 + \beta^2\right).$$

where $T_2 = \min\{T_1, \beta\}$. By the definition of a, we conclude that

$$t^2|\nabla^2 g|^2 \leq c_{11}\beta$$

provided $\beta \leq 1$. Now choose $\beta \leq 1$ such that $c_{11}\beta < \alpha^2$. β depends only on n, α. Choose choose b satisfying (**c1**) and a satisfying (**c3**). If $t \leq \beta, T_1$, then we have

$$t^2|\nabla^2 g|^2 \leq \alpha^2$$

in $M\times[0, T_2]$, where $T_2 = \min\{T_1, \beta\}$. This completes the proof of the lemma. □

Lemma 5.3. *For any $\alpha > 0$, there exists $\epsilon(n, \alpha) > 0$ depending only on n and α such that if (M^n, g_0) is a complete noncompact Riemannian manifold with real dimension n and if g_0 is $(1 + \epsilon)$ close to a Riemannian metric h with curvature bounded by k_0, then there is a smooth complete Ricci flow $g(t)$ defined on $M \times [0, T]$ with initial value $g(0) = g_0$, where $T > 0$ depends only*

on n, k_0. Moreover, there is $T_1(n, k_0, \alpha) > 0$ depending only on n, α such that the curvature of $g(t)$ satisfies:

$$|\mathrm{Rm}(g(t))|_{g(t)} \le \frac{\alpha}{t}$$

on $M \times [0, T_1]$.

Proof. First we remark that by [20], there is a solution to the Ricci flow with initial data h with bounded curvature in space and time. Moreover, for $t > 0$ all order of derivatives of the curvature tensor for a fixed $t > 0$ are uniformly bounded, the solution exists in a time interval depending only on n, k_0, and the bounds of the derivatives of the curvature tensor for a fixed $t > 0$ depend only on n, k_0, and t. Hence without lost of generality, we may assume that $|\widetilde{\nabla}^{(i)}\widetilde{\mathrm{Rm}}|_h \le k_i < \infty$ for all $i \ge 0$. Here and in the following $\widetilde{\nabla}$ is the covariant derivative with respect to h and $\widetilde{\mathrm{Rm}}$ is the curvature tensor of h and $|\cdot|_h$ is the norm relative to h.

Note that if h is $1+\epsilon$ close to g_0, then λh is also $1+\epsilon$ close to λg_0 for any $\lambda > 0$. Moreover, if $g(t)$ is a solution to the Ricci flow with initial data g_0, then $\lambda g(\lambda^{-1}t)$ is a solution to the Ricci flow with initial data λg_0, and if $s = \lambda t$, then

$$|\mathrm{Rm}(g(t))|_{g(t)} = \lambda |\mathrm{Rm}(\lambda g(\lambda^{-1}s))|_{\lambda g(\lambda^{-1}s)}.$$

Hence we may assume that $k_0 + k_1 + k_2 \le 1$.

Let us first assume that $\epsilon(n, \alpha) < \epsilon(n)$ where $\epsilon(n)$ is the constant in Theorem 5.1.

For any $R > 0$, let $0 \le \eta_R$ be a smooth function on M such that

$$\eta_R = \begin{cases} 1, & x \in B_{g_0}(x_0, R) \\ 0, & x \in M \setminus B_{g_0}(x_0, 2R) \end{cases}$$

Let $g_{R,0} = \eta_R g_0 + (1 - \eta_R)h$. Let $\beta > 0$ to be chosen later depending only on n, α. Suppose h is $e^b = (1+\epsilon)$ close to g_0, where $b > 0$, then one can see that h is also e^b close to $g_{R,0}$ for all $R > 0$. By Lemmas 5.1, 5.2, there is a constant $b > 0$ depending only on n, β such that for all $R > 0$ the solution $\bar{g}_R(t)$ to the h-flow as in Theorem 5.1 exists on $M \times [0, T_2]$ for some $T_2 > 0$ depending only on n, β such that

$$|\widetilde{\nabla}^i \bar{g}_R(t)|_h^2 \le \frac{C_i}{t^i} \tag{5.12}$$

for all $i \ge 0$, where C_i depends only on $n, i, k_0, \ldots, k_i$. Moreover,

$$|\bar{g}_R(t)|_h \le 2, |\widetilde{\nabla}^i \bar{g}_R(t)|_h^2| \le \frac{\beta^i}{t^i} \ \text{ for } i = 1, 2. \tag{5.13}$$

Now we want to claim that there is a constant $c_1 = c_1(n)$ depending only on n such that

$$|\mathrm{Rm}(\bar{g}_R(t))|_{g_R(t)} \le c_1 \left(|\widetilde{\mathrm{Rm}}|_h + |\widetilde{\nabla} \bar{g}_R(t)|_h^2 + |\widetilde{\nabla}^2 \bar{g}_R(t)|_h \right). \tag{5.14}$$

To see (5.14), we choose a normal coordinate at any fix point $x \in M$ with respect to h and it also diagonalizes $\bar g_R(t)$. In this coordinate, we have

$$\begin{aligned}
\bar R^l_{ijk} =& \frac{\partial}{\partial x^i}\bar\Gamma^l_{kj} - \frac{\partial}{\partial x^j}\bar\Gamma^l_{ki} + \bar\Gamma^h_{kj}\bar\Gamma^l_{hi} - \bar\Gamma^h_{ki}\bar\Gamma^l_{hj}\\
=& \frac{\partial}{\partial x^i}(\bar\Gamma^l_{kj} - \widetilde\Gamma^l_{kj}) - \frac{\partial}{\partial x^j}(\bar\Gamma^l_{ki} - \widetilde\Gamma^l_{ki})\\
&+ \frac{\partial}{\partial x^i}\widetilde\Gamma^l_{kj} - \frac{\partial}{\partial x^j}\widetilde\Gamma^l_{ki} + \bar\Gamma^h_{kj}\bar\Gamma^l_{hi} - \bar\Gamma^h_{ki}\bar\Gamma^l_{hj}\\
=& \widetilde R^l_{kij} + \widetilde\nabla_i(\bar\Gamma^l_{kj} - \widetilde\Gamma^l_{kj}) - \widetilde\nabla_j(\bar\Gamma^l_{ki} - \widetilde\Gamma^l_{ki}) + \bar\Gamma^h_{kj}\bar\Gamma^l_{hi} - \bar\Gamma^h_{ki}\bar\Gamma^l_{hj}.
\end{aligned}$$

Here we use $\bar\cdot$ to denote the Christoffel symbol and curvature tensor of the metric $\bar g_R(t)$. Note that

$$\bar\Gamma^l_{kj} - \widetilde\Gamma^l_{kj} = \frac{1}{2}g^{ls}(\widetilde\nabla_k g_{js} + \widetilde\nabla_j g_{ks} - \widetilde\nabla_s g_{kj}),$$

we have

$$\widetilde\nabla_i(\bar\Gamma^l_{kj} - \widetilde\Gamma^l_{kj}) = g^{-1} * g^{-1}\widetilde\nabla g * \widetilde\nabla g + g^{-1} * \widetilde\nabla^2 g$$

and

$$\bar\Gamma * \bar\Gamma = g^{-1} * g^{-1} * \widetilde\nabla g * \widetilde\nabla g.$$

Therefore, we obtain (5.14).

Then, we have

$$|\mathrm{Rm}(\bar g_R(t))|_{g_R(t)} \le \frac{3c_1\beta}{t} \tag{5.15}$$

provided on $M\times[0,T_2]$ provided T_2 is small depending only on n, β, k_0. Here we have used the fact that h is $1+2\epsilon(n)$ close to $\bar g_R(t)$.

Using the similar argument as in the proof of [18, Lemma 4.1] or Lemma 3.1, one can show that

$$|\widetilde\nabla \bar g_R(t)| \le C(n, h, g_0, R)$$

for some constant $C(n,h,g_0,R)$ depending only on n,h,g_0,R. Hence we can pull back $\bar g_R(t)$ by a smooth family of diffeomorphisms from M to itself $\varphi_R(t)$, $t\in[0,T]$. That is, let $g_R(t) = \varphi_R(t)^*\bar g_R(t)$ on $M\times[0,T_2]$ where $\varphi_R(t)$, $t\in[0,T_2]$ is given by solving the following ODE at each point $x\in M$:

$$\begin{cases} \frac{d}{dt}\varphi_R(x,t) = -W(\varphi_R(x,t),t) \\ \varphi_R(x,0) = x \end{cases} \tag{5.16}$$

where W is a time-dependent smooth vector field given by

$$W^i(t) = \bar g_R^{jk}(t)({}^{\bar g_R(t)}\Gamma^i_{jk} - {}^h\Gamma^i_{jk}).$$

Then $g_R(t)$ is a solution to the Ricci flow with $g_R(0) = g_{R,0}$. By (5.15)

$$|Rm(g_R(t))|_{g_R(t)} \le \frac{3c_1\beta}{t}$$

on $M \times (0, T_2]$. Since $g_{R,0}$ has uniformly bounded curvature, which may depends on R, by [6, 19] for any compact set U, there is a constant C_1 independent of R such that

$$|Rm(g_R(t))|_{g_R(t)} \le C_1$$

on $U \times [0, T_2]$. By [20] (see also [13, Theorem 11]), we see that for each m, there is a constant $C(m)$ independent of R such that

$$|^{g_R(t)}\nabla^m Rm_{g_R(t)}|_{g_R(t)} \le C(m)$$

on $U \times [0, T_2]$. From this, we obtain that

$$|^{g_R(t)}\nabla^m g_R(t)|_{g_R} \le C(m)$$

on $U \times [0, T_2]$ for some constant $C(m)$ independent of R. Hence by diagonal process, passing to a subsequence, $g_R(t)$ converges in C^∞ topology on compact sets of $M \times [0, T_2]$ to a solution $g(t)$ of the Ricci flow with $g(0) = g_0$. Moreover, by (5.15),

$$|\mathrm{Rm}(g(t))|_{g(t)} \le \frac{3c_1\beta}{t}.$$

Next, we claim $g(t)$ is complete for all $t \in [0, T_2]$. Let $\{y_k\}$ be a divergence sequence of points in M. For any fixed point x_0 and $t \in [0, T_2]$, we have

$$\begin{aligned} d_{g_R(t)}(x_0, y_k) =& d_{\bar{g}_R(t)}(\varphi_R(x_0, t), \varphi_R(y_k, t)) \\ &\ge d_{\bar{g}_R(t)}(x_0, y_k) - d_{\bar{g}_R(t)}(\varphi_R(x_0, t), x_0) - d_{\bar{g}_R(t)}(y_k, \varphi_R(y_k, t)), \end{aligned}$$

for some positive constants C_3, C_4 independent of R, y_k, where we have used (5.12) which implies $W(x, t)$ in (5.16) is uniformly bounded by a $Ct^{-\frac{1}{2}}$ for some constant C for all x, t and R, and we have also used the fact that $(1+2\epsilon)^{-1}h \le g_R \le (1 + 2\epsilon)h$ for all R and $t \in [0, T]$. This implies

$$d_{g_R(t)}(x_0, y_n) \ge C_3 d_h(x_0, y_n) - C_4\sqrt{T_2},$$

Let $R \to +\infty$, we see that

$$d_{g(t)}(x_0, y_k) \ge C_3 d_h(x_0, y_n) - C_4\sqrt{T_2}.$$

Since h is complete, we obtain $d_{g(t)}(x_0, y_k) \to \infty$ as $k \to \infty$. This implies $g(t)$ is complete.

Now choose β such that $3c_1\beta = \alpha$, we conclude that the lemma is true. □

Now we want to prove the main result of this section:

Theorem 5.2. *There exists $\epsilon(2n) > 0$ depending only on n such that if (M^n, g_0) is a complete noncompact Kähler manifold with complex dimension n and if there is a smooth Riemannian metric h with curvature bounded by k_0 on M such that g_0 is $(1+\epsilon(n))$ close h, then there is a complete Kähler-Ricci flow $g(t)$ defined on $M \times [0, T]$ with initial value $g(0) = g_0$, where $T > 0$ depends only on n, k_0. Moreover, the curvature of $g(t)$ satisfies:*

$$|\mathrm{Rm}(g(t))|_{g(t)} \le \frac{\alpha}{t}$$

where $\alpha = \alpha(n)$ is the constant in Theorem 4.1. If in addition, g_0 has nonnegative holomorphic bisectional curvature, then $g(t)$ has nonnegative holomorphic bisectional curvature for all $t \in [0, T]$.

Proof. The results follow from Lemma 5.3 and Theorems 3.1, 4.1. □

Corollary 5.1. *Let $\epsilon(2n)$ be as in Theorem 5.2. Suppose (M^n, g_0) is a complete noncompact Kähler manifold with complex dimension n with nonnegative holomorphic bisectional curvature with maximum volume growth. Suppose there is a Riemannian metric h on M with bounded curvature which is $1+\epsilon(2n)$ close to g_0. Then M is biholomorphic to $\mathbb{C}^n$.*

Proof. Let $g(t)$ be the solution of Kähler-Ricci flow obtained in Theorem 5.2. Then for $t > 0$, $g(t)$ is Kähler with bounded nonnegative holomorphic bisectional curvature. We claim that $g(t)$ has maximum volume growth. Let $x_0 \in M$ be fixed. By the proof of Lemma 5.3, using the same notations as in the proof we conclude that

$$V_{\bar{g}_R(t)}(x_0, r) \geq C_1 r^{2n} \tag{5.17}$$

for some $C_1 > 0$ for all r because g_0 has maximum volume growth and $\bar{g}_R(t)$ is uniformly equivalent to h which in turn is uniformly equivalent to g_0. Here $V_{\bar{g}_R(t)}(x_0, r)$ is the volume of the geodesic ball $B_{\bar{g}_R(t)}(x_0, r)$ with respect to $\bar{g}_R(t)$. As in the proof of Lemma 5.3,

$$V_{\bar{g}_R(t)}(x_0, r) = V_{g_R(t)}(\varphi_t^{-1}(x_0), r) \leq V_{g_R(t)}(x_0, r + C_2)$$

for some constant $C_2 > 0$ independent of R and x_0. From this and (5.17), we conclude that $g(t)$ has maximum volume growth. Hence M is biholomorphic to $\mathbb{C}^n$ by the result of [5].

□

References

[1] Bando, S.,*On the classification of three-dimensional compact Kaehler manifolds of nonnegative bisectional curvature*, J. Differential Geom. **19** (1984), no. 2, 283–297.

[2] Cabezas-Rivas, E.; Wilking, B., *How to produce a Ricci Flow via Cheeger-Gromoll exhaustion* arXiv:1107.0606 (2011)

[3] Chau, A.; Li, K.-F.; Tam, L.-F., *Deforming complete Hermitian metrics with unbounded curvature*, to appear in Asian J. Math, arXiv:1402.6722

[4] Chau, A.; Li, K.-F.; Tam, L.-F., *Longtime existence of the Khler-Ricci flow on $\mathbb{C}^n$*, arXiv:1409.1906.

[5] Chau, A. ; Tam, L.-F., *On the complex structure of Kähler manifolds with nonnegative curvature*, J. Differential Geom. **73** (2006), 491–530, MR 2228320

[6] Chen, B.L., *Strong Uniqueness of the Ricci Flow*, J. Differential Geom. 82 (2009), no. 2, 363–382.

[7] Chen, B.L.; Tang, S.H.; and Zhu, X.P., *A Uniformization Theorem Of Complete Noncompact Kähler Surfaces With Positive Bisectional Curvature*, J. Differential Geom. 67 (2004), no. 3, 519570.

[8] Chow, B.; Chu, S.-C.; Glickenstein, D.; Guenther, C.; Isenberg, J.; Ivey, T.; Knopf, D.; Lu, P.; Luo, F.; Ni, L., *The Ricci flow : techniques and applications, Part III. Geometric-analytic aspects* Mathematical Surveys and Monographs, **135**, American Mathematical Society, Providence, RI, (2007)

[9] Chow, B.; Knopf, D., *The Ricci flow: an introduction. Mathematical Surveys and Monographs* **110**, American Mathematical Society, 2004.

[10] G. Giesen; P.M. Topping, *Existence of Ricci flows of incomplete surfaces*, Comm. Partial Differential Equations **36** (2011), no. 10, 1860–1880.

[11] Hamilton, R. S., *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.

[12] Liu, G.,*Gromov-Hausdorff limit of Khler manifolds and the finite generation conjecture*, arXiv:1501.00681.

[13] Lu, P.; Tian, G., *Uniqueness of standard solutions in the work of Perelman* Available at www.math.lsa.umich.edu/ lott/ricciflow/StanUniqWork2.pdf

[14] Mok, N., *An embedding theorem of complete Kähler manifolds of positive bisectional curvature onto affine algebraic varieties*, Bull. Soc. Math. France **112** (1984), no. 2, 197–250.

[15] Mok, N., *The Uniformization Theorem for Compact Kähler manifolds of Nonnegative Holomorphic Bisectional Curvature* J. Differential Geometry, **27** (1988) 179–214

[16] Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159

[17] Petrunin, A. M., *An upper bound for the curvature integral*, Algebra i Analiz **20** (2008), no. 2, 134–148; translation in St. Petersburg Math. J. **20** (2009), no. 2, 255–265

[18] Simon, M., *Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature*, Comm. Anal. Geom. **10**(2002), no. 5, 1033–1074

[19] Simon, M., *Local results for flows whose speed or height is bounded by c/t*, Int. Math. Res. Not. IMRN (2008), Art. ID rnn **097**, 14 pp

[20] Shi, W.-X., *Deforming the metric on complete Riemannian manifolds*, J. Differential Geom. 30 (1989), no. 1, 223–301

[21] Shi, W.-X., *Ricci Flow and the uniformization on complete non compact Kähler manifolds*, J. of Differential Geometry **45** (1997), 94–220

[22] Tam, L.-F., *Exhaustion functions on complete manifolds*, 211–215 in Recent advances in geometric analysis, Adv. Lect. Math. (ALM), **11**, Int. Press, Somerville, MA, 2010

[23] Xu, G., *Short-time existence of the Ricci flow on noncompact Riemannian manifolds*, Trans. Amer. Math. Soc. **365** (2013), no. 11, 5605–5654.

[24] Yang, B.; Zheng, F.-Y., $U(n)$-invariant Kähler-Ricci flow with non-negative curvature, Comm. Anal. Geom. **21** (2013), no. 2, 251–294.

[25] Yau, S.-T., *A review of complex differential geometry*, Proc. Sympos. Pure Math., **52** Part 2 (1991), 619–625.

[26] Zheng, F.-Y.,*First Pontrjagin form, rigidity and strong rigidity of nonpositively curved Kähler surface of general type*, Math. Z. **220** (1995), no. 2, 159–169.

Department of Mathematics, The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China.

E-mail address: `schuang@math.cuhk.edu.hk`

The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China.

E-mail address: `lftam@math.cuhk.edu.hk`